Math 21C
Kouba
The Sequence of Partial Sums Test

RECALL: A sequence \( \{a_n\} \) is a function which assigns a real number \( a_n \) to each natural number \( n: 1, 2, 3, 4, 5, \ldots \), i.e., a sequence is an ordered list of real numbers: \( a_1, a_2, a_3, a_4, a_5, \ldots \).

EXAMPLE: \( \left\{ \frac{2^n - 1}{n + 7} \right\} \) generates the sequence \( \frac{1}{8}, \frac{2}{9}, \frac{4}{10}, \frac{8}{11}, \frac{16}{12}, \ldots \).

DEFINITION: An infinite series \( \sum_{n=1}^{\infty} a_n \) is the sum of the numbers in the sequence \( \{a_n\} \), i.e., \( \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots \).

EXAMPLE: \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \).

But how do we add an infinite number of numbers together? We need a more precise definition of an infinite series \( \sum_{n=1}^{\infty} a_n \). Begin by constructing a new sequence of partial sums by letting (This step by step process will be called the Sequence of Partial Sums Test for the infinite series \( \sum_{n=1}^{\infty} a_n \).)

\[
\begin{align*}
s_1 &= a_1, \\
s_2 &= a_1 + a_2, \\
s_3 &= a_1 + a_2 + a_3, \\
s_4 &= a_1 + a_2 + a_3 + a_4, \\
\vdots & \quad \vdots \\
s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n.
\end{align*}
\]

We can now say that the value of the infinite series is precisely the value of the limit of its sequence of partial sums, i.e.,

\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots = \lim_{n \to \infty} (a_1 + a_2 + a_3 + a_4 + \cdots + a_n) = \lim_{n \to \infty} s_n.
\]