Math 21C
Kouba
Finding the Second Partial Derivative Using the Chain Rule

Assume that we are given the functions \( z = f(x, y) \), \( x = g(s, t) \), and \( y = k(s, t) \). Our goal is to determine the form of the second partial derivative of \( z \) with respect to \( t \), \( \frac{\partial^2 z}{\partial t^2} \). (In a similar fashion we can determine \( \frac{\partial^2 z}{\partial s^2} \).) We will use the diagrams on the right to guide us. The first partial derivative of \( z \) with respect to \( t \) is

\[
\frac{\partial z}{\partial t} = z_x \cdot \frac{\partial x}{\partial t} + z_y \cdot \frac{\partial y}{\partial t}.
\]

The second partial derivative is now

\[
\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial t} \right) = \frac{\partial}{\partial t} \left( z_x \cdot \frac{\partial x}{\partial t} + z_y \cdot \frac{\partial y}{\partial t} \right)
\]

(Use the Product Rule twice and again use the Chain Rule twice.)

\[
= \left\{ z_x \cdot \frac{\partial}{\partial t} \left( \frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} (z_x) \cdot \frac{\partial x}{\partial t} \right\} + \left\{ z_y \cdot \frac{\partial}{\partial t} \left( \frac{\partial y}{\partial t} \right) + \frac{\partial}{\partial t} (z_y) \cdot \frac{\partial y}{\partial t} \right\}
\]

\[
= z_x \cdot \frac{\partial^2 x}{\partial t^2} + \left[ z_{xx} \cdot \frac{\partial x}{\partial t} + z_{xy} \cdot \frac{\partial y}{\partial t} \right] \cdot \frac{\partial x}{\partial t}
\]

\[
+ z_y \cdot \frac{\partial^2 y}{\partial t^2} + \left[ z_{yx} \cdot \frac{\partial x}{\partial t} + z_{yy} \cdot \frac{\partial y}{\partial t} \right] \cdot \frac{\partial y}{\partial t}
\]

\[
= z_x \cdot \frac{\partial^2 x}{\partial t^2} + z_y \cdot \frac{\partial^2 y}{\partial t^2} + z_{xx} \cdot \left( \frac{\partial x}{\partial t} \right)^2
\]

\[
+ 2z_{xy} \cdot \left( \frac{\partial x}{\partial t} \right) \left( \frac{\partial y}{\partial t} \right) + z_{yy} \cdot \left( \frac{\partial y}{\partial t} \right)^2.
\]