lowest such line. None of the points (n, a_n) lies above $y = L$, but some do lie above any lower line $y = L - \epsilon$, if ϵ is a positive number. The sequence converges to L because

a. $a_n \le L$ for *all* values of n, and

b. given any $\epsilon > 0$, there exists at least one integer N for which $a_N > L - \epsilon$.

The fact that $\{a_n\}$ is nondecreasing tells us further that

$$a_n \ge a_N > L - \epsilon \qquad \text{for all } n \ge N.$$

Thus, *all* the numbers a_n beyond the Nth number lie within ϵ of L. This is precisely the condition for L to be the limit of the sequence $\{a_n\}$.

The proof for nonincreasing sequences bounded from below is similar. ■

It is important to realize that Theorem 6 does not say that convergent sequences are monotonic. The sequence $\{(-1)^{n+1}/n\}$ converges and is bounded, but it is not monotonic since it alternates between positive and negative values as it tends toward zero. What the theorem does say is that a nondecreasing sequence converges when it is bounded from above, but it diverges to infinity otherwise.

Exercises 10.1

Finding Terms of a Sequence

Each of Exercises 1–6 gives a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3, and a_4.

1. $a_n = \dfrac{1 - n}{n^2}$
2. $a_n = \dfrac{1}{n!}$
3. $a_n = \dfrac{(-1)^{n+1}}{2n - 1}$
4. $a_n = 2 + (-1)^n$
5. $a_n = \dfrac{2^n}{2^{n+1}}$
6. $a_n = \dfrac{2^n - 1}{2^n}$

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

7. $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$
8. $a_1 = 1, \quad a_{n+1} = a_n/(n + 1)$
9. $a_1 = 2, \quad a_{n+1} = (-1)^{n+1}a_n/2$
10. $a_1 = -2, \quad a_{n+1} = na_n/(n + 1)$
11. $a_1 = a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$
12. $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = a_{n+1}/a_n$

Finding a Sequence's Formula

In Exercises 13–26, find a formula for the nth term of the sequence.

13. The sequence $1, -1, 1, -1, 1, \ldots$ — 1's with alternating signs
14. The sequence $-1, 1, -1, 1, -1, \ldots$ — 1's with alternating signs
15. The sequence $1, -4, 9, -16, 25, \ldots$ — Squares of the positive integers, with alternating signs
16. The sequence $1, -\dfrac{1}{4}, \dfrac{1}{9}, -\dfrac{1}{16}, \dfrac{1}{25}, \ldots$ — Reciprocals of squares of the positive integers, with alternating signs
17. $\dfrac{1}{9}, \dfrac{2}{12}, \dfrac{2^2}{15}, \dfrac{2^3}{18}, \dfrac{2^4}{21}, \ldots$ — Powers of 2 divided by multiples of 3
18. $-\dfrac{3}{2}, -\dfrac{1}{6}, \dfrac{1}{12}, \dfrac{3}{20}, \dfrac{5}{30}, \ldots$ — Integers differing by 2 divided by products of consecutive integers
19. The sequence $0, 3, 8, 15, 24, \ldots$ — Squares of the positive integers diminished by 1
20. The sequence $-3, -2, -1, 0, 1, \ldots$ — Integers, beginning with -3
21. The sequence $1, 5, 9, 13, 17, \ldots$ — Every other odd positive integer
22. The sequence $2, 6, 10, 14, 18, \ldots$ — Every other even positive integer
23. $\dfrac{5}{1}, \dfrac{8}{2}, \dfrac{11}{6}, \dfrac{14}{24}, \dfrac{17}{120}, \ldots$ — Integers differing by 3 divided by factorials
24. $\dfrac{1}{25}, \dfrac{8}{125}, \dfrac{27}{625}, \dfrac{64}{3125}, \dfrac{125}{15{,}625}, \ldots$ — Cubes of positive integers divided by powers of 5
25. The sequence $1, 0, 1, 0, 1, \ldots$ — Alternating 1's and 0's
26. The sequence $0, 1, 1, 2, 2, 3, 3, 4, \ldots$ — Each positive integer repeated

Convergence and Divergence

Which of the sequences $\{a_n\}$ in Exercises 27–90 converge, and which diverge? Find the limit of each convergent sequence.

27. $a_n = 2 + (0.1)^n$
28. $a_n = \dfrac{n + (-1)^n}{n}$
29. $a_n = \dfrac{1 - 2n}{1 + 2n}$
30. $a_n = \dfrac{2n + 1}{1 - 3\sqrt{n}}$
31. $a_n = \dfrac{1 - 5n^4}{n^4 + 8n^3}$
32. $a_n = \dfrac{n + 3}{n^2 + 5n + 6}$
33. $a_n = \dfrac{n^2 - 2n + 1}{n - 1}$
34. $a_n = \dfrac{1 - n^3}{70 - 4n^2}$

35. $a_n = 1 + (-1)^n$

36. $a_n = (-1)^n\left(1 - \frac{1}{n}\right)$

37. $a_n = \left(\frac{n+1}{2n}\right)\left(1 - \frac{1}{n}\right)$

38. $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right)$

39. $a_n = \frac{(-1)^{n+1}}{2n - 1}$

40. $a_n = \left(-\frac{1}{2}\right)^n$

41. $a_n = \sqrt{\frac{2n}{n+1}}$

42. $a_n = \frac{1}{(0.9)^n}$

43. $a_n = \sin\left(\frac{\pi}{2} + \frac{1}{n}\right)$

44. $a_n = n\pi \cos(n\pi)$

45. $a_n = \frac{\sin n}{n}$

46. $a_n = \frac{\sin^2 n}{2^n}$

47. $a_n = \frac{n}{2^n}$

48. $a_n = \frac{3^n}{n^3}$

49. $a_n = \frac{\ln(n+1)}{\sqrt{n}}$

50. $a_n = \frac{\ln n}{\ln 2n}$

51. $a_n = 8^{1/n}$

52. $a_n = (0.03)^{1/n}$

53. $a_n = \left(1 + \frac{7}{n}\right)^n$

54. $a_n = \left(1 - \frac{1}{n}\right)^n$

55. $a_n = \sqrt[n]{10n}$

56. $a_n = \sqrt[n]{n^2}$

57. $a_n = \left(\frac{3}{n}\right)^{1/n}$

58. $a_n = (n+4)^{1/(n+4)}$

59. $a_n = \frac{\ln n}{n^{1/n}}$

60. $a_n = \ln n - \ln(n+1)$

61. $a_n = \sqrt[n]{4^n n}$

62. $a_n = \sqrt[n]{3^{2n+1}}$

63. $a_n = \frac{n!}{n^n}$ (*Hint:* Compare with $1/n$.)

64. $a_n = \frac{(-4)^n}{n!}$

65. $a_n = \frac{n!}{10^{6n}}$

66. $a_n = \frac{n!}{2^n \cdot 3^n}$

67. $a_n = \left(\frac{1}{n}\right)^{1/(\ln n)}$

68. $a_n = \ln\left(1 + \frac{1}{n}\right)^n$

69. $a_n = \left(\frac{3n+1}{3n-1}\right)^n$

70. $a_n = \left(\frac{n}{n+1}\right)^n$

71. $a_n = \left(\frac{x^n}{2n+1}\right)^{1/n}, \quad x > 0$

72. $a_n = \left(1 - \frac{1}{n^2}\right)^n$

73. $a_n = \frac{3^n \cdot 6^n}{2^{-n} \cdot n!}$

74. $a_n = \frac{(10/11)^n}{(9/10)^n + (11/12)^n}$

75. $a_n = \tanh n$

76. $a_n = \sinh(\ln n)$

77. $a_n = \frac{n^2}{2n - 1}\sin\frac{1}{n}$

78. $a_n = n\left(1 - \cos\frac{1}{n}\right)$

79. $a_n = \sqrt{n}\sin\frac{1}{\sqrt{n}}$

80. $a_n = (3^n + 5^n)^{1/n}$

81. $a_n = \tan^{-1} n$

82. $a_n = \frac{1}{\sqrt{n}}\tan^{-1} n$

83. $a_n = \left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}}$

84. $a_n = \sqrt[n]{n^2 + n}$

85. $a_n = \frac{(\ln n)^{200}}{n}$

86. $a_n = \frac{(\ln n)^5}{\sqrt{n}}$

87. $a_n = n - \sqrt{n^2 - n}$

88. $a_n = \frac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}}$

89. $a_n = \frac{1}{n}\int_1^n \frac{1}{x}\,dx$

90. $a_n = \int_1^n \frac{1}{x^p}\,dx, \quad p > 1$

Recursively Defined Sequences

In Exercises 91–98, assume that each sequence converges and find its limit.

91. $a_1 = 2, \quad a_{n+1} = \frac{72}{1 + a_n}$

92. $a_1 = -1, \quad a_{n+1} = \frac{a_n + 6}{a_n + 2}$

93. $a_1 = -4, \quad a_{n+1} = \sqrt{8 + 2a_n}$

94. $a_1 = 0, \quad a_{n+1} = \sqrt{8 + 2a_n}$

95. $a_1 = 5, \quad a_{n+1} = \sqrt{5a_n}$

96. $a_1 = 3, \quad a_{n+1} = 12 - \sqrt{a_n}$

97. $2, 2 + \frac{1}{2}, 2 + \frac{1}{2 + \frac{1}{2}}, 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \ldots$

98. $\sqrt{1}, \sqrt{1 + \sqrt{1}}, \sqrt{1 + \sqrt{1 + \sqrt{1}}}, \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}, \ldots$

Theory and Examples

99. The first term of a sequence is $x_1 = 1$. Each succeeding term is the sum of all those that come before it:

$$x_{n+1} = x_1 + x_2 + \cdots + x_n.$$

Write out enough early terms of the sequence to deduce a general formula for x_n that holds for $n \geq 2$.

100. A sequence of rational numbers is described as follows:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{a}{b}, \frac{a + 2b}{a + b}, \ldots.$$

Here the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let x_n and y_n be, respectively, the numerator and the denominator of the nth fraction $r_n = x_n/y_n$.

a. Verify that $x_1^2 - 2y_1^2 = -1, x_2^2 - 2y_2^2 = +1$ and, more generally, that if $a^2 - 2b^2 = -1$ or $+1$, then

$$(a + 2b)^2 - 2(a + b)^2 = +1 \quad \text{or} \quad -1,$$

respectively.

b. The fractions $r_n = x_n/y_n$ approach a limit as n increases. What is that limit? (*Hint:* Use part (a) to show that $r_n^2 - 2 = \pm(1/y_n)^2$ and that y_n is not less than n.)

101. Newton's method The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Do the sequences converge? If so, to what value? In each case, begin by identifying the function f that generates the sequence.

a. $x_0 = 1, \quad x_{n+1} = x_n - \dfrac{x_n^2 - 2}{2x_n} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$

b. $x_0 = 1, \quad x_{n+1} = x_n - \dfrac{\tan x_n - 1}{\sec^2 x_n}$

c. $x_0 = 1, \quad x_{n+1} = x_n - 1$

102. a. Suppose that $f(x)$ is differentiable for all x in $[0, 1]$ and that $f(0) = 0$. Define sequence $\{a_n\}$ by the rule $a_n = nf(1/n)$. Show that $\lim_{n\to\infty} a_n = f'(0)$. Use the result in part (a) to find the limits of the following sequences $\{a_n\}$.

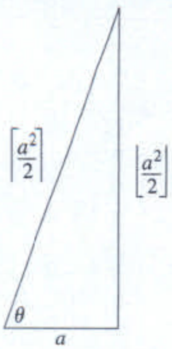
b. $a_n = n \tan^{-1} \dfrac{1}{n}$ c. $a_n = n(e^{1/n} - 1)$

d. $a_n = n \ln\left(1 + \dfrac{2}{n}\right)$

103. **Pythagorean triples** A triple of positive integers a, b, and c is called a **Pythagorean triple** if $a^2 + b^2 = c^2$. Let a be an odd positive integer and let

$$b = \left\lfloor \frac{a^2}{2} \right\rfloor \quad \text{and} \quad c = \left\lceil \frac{a^2}{2} \right\rceil$$

be, respectively, the integer floor and ceiling for $a^2/2$.



a. Show that $a^2 + b^2 = c^2$. (*Hint:* Let $a = 2n + 1$ and express b and c in terms of n.)

b. By direct calculation, or by appealing to the accompanying figure, find

$$\lim_{a\to\infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil}.$$

104. **The nth root of $n!$**

a. Show that $\lim_{n\to\infty}(2n\pi)^{1/(2n)} = 1$ and hence, using Stirling's approximation (Chapter 8, Additional Exercise 52a), that

$$\sqrt[n]{n!} \approx \frac{n}{e} \quad \text{for large values of } n.$$

T b. Test the approximation in part (a) for $n = 40, 50, 60, \ldots$, as far as your calculator will allow.

105. a. Assuming that $\lim_{n\to\infty}(1/n^c) = 0$ if c is any positive constant, show that

$$\lim_{n\to\infty} \frac{\ln n}{n^c} = 0$$

if c is any positive constant.

b. Prove that $\lim_{n\to\infty}(1/n^c) = 0$ if c is any positive constant. (*Hint:* If $\epsilon = 0.001$ and $c = 0.04$, how large should N be to ensure that $|1/n^c - 0| < \epsilon$ if $n > N$?)

106. **The zipper theorem** Prove the "zipper theorem" for sequences: If $\{a_n\}$ and $\{b_n\}$ both converge to L, then the sequence

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$$

converges to L.

107. Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

108. Prove that $\lim_{n\to\infty} x^{1/n} = 1, (x > 0)$.

109. Prove Theorem 2.

110. Prove Theorem 3.

In Exercises 111–114, determine if the sequence is monotonic and if it is bounded.

111. $a_n = \dfrac{3n + 1}{n + 1}$

112. $a_n = \dfrac{(2n + 3)!}{(n + 1)!}$

113. $a_n = \dfrac{2^n 3^n}{n!}$

114. $a_n = 2 - \dfrac{2}{n} - \dfrac{1}{2^n}$

Which of the sequences in Exercises 115–124 converge, and which diverge? Give reasons for your answers.

115. $a_n = 1 - \dfrac{1}{n}$

116. $a_n = n - \dfrac{1}{n}$

117. $a_n = \dfrac{2^n - 1}{2^n}$

118. $a_n = \dfrac{2^n - 1}{3^n}$

119. $a_n = ((-1)^n + 1)\left(\dfrac{n + 1}{n}\right)$

120. The first term of a sequence is $x_1 = \cos(1)$. The next terms are $x_2 = x_1$ or $\cos(2)$, whichever is larger; and $x_3 = x_2$ or $\cos(3)$, whichever is larger (farther to the right). In general,

$$x_{n+1} = \max\{x_n, \cos(n + 1)\}.$$

121. $a_n = \dfrac{1 + \sqrt{2n}}{\sqrt{n}}$

122. $a_n = \dfrac{n + 1}{n}$

123. $a_n = \dfrac{4^{n+1} + 3^n}{4^n}$

124. $a_1 = 1, \quad a_{n+1} = 2a_n - 3$

In Exercises 125–126, use the definition of convergence to prove the given limit.

125. $\lim_{n\to\infty} \dfrac{\sin n}{n} = 0$

126. $\lim_{n\to\infty} \left(1 - \dfrac{1}{n^2}\right) = 1$

127. **The sequence $\{n/(n + 1)\}$ has a least upper bound of 1** Show that if M is a number less than 1, then the terms of $\{n/(n + 1)\}$ eventually exceed M. That is, if $M < 1$ there is an integer N such that $n/(n + 1) > M$ whenever $n > N$. Since $n/(n + 1) < 1$ for every n, this proves that 1 is a least upper bound for $\{n/(n + 1)\}$.

128. **Uniqueness of least upper bounds** Show that if M_1 and M_2 are least upper bounds for the sequence $\{a_n\}$, then $M_1 = M_2$. That is, a sequence cannot have two different least upper bounds.

129. Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.

130. Prove that if $\{a_n\}$ is a convergent sequence, then to every positive number ϵ there corresponds an integer N such that for all m and n,

$$m > N \quad \text{and} \quad n > N \quad \Rightarrow \quad |a_m - a_n| < \epsilon.$$

131. Uniqueness of limits Prove that limits of sequences are unique. That is, show that if L_1 and L_2 are numbers such that $a_n \to L_1$ and $a_n \to L_2$, then $L_1 = L_2$.

132. Limits and subsequences If the terms of one sequence appear in another sequence in their given order, we call the first sequence a **subsequence** of the second. Prove that if two subsequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges.

133. For a sequence $\{a_n\}$ the terms of even index are denoted by a_{2k} and the terms of odd index by a_{2k+1}. Prove that if $a_{2k} \to L$ and $a_{2k+1} \to L$, then $a_n \to L$.

134. Prove that a sequence $\{a_n\}$ converges to 0 if and only if the sequence of absolute values $\{|a_n|\}$ converges to 0.

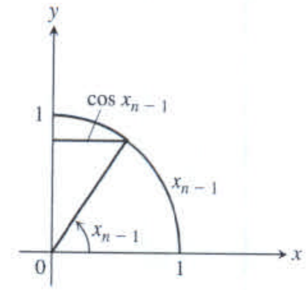
135. Sequences generated by Newton's method Newton's method, applied to a differentiable function $f(x)$, begins with a starting value x_0 and constructs from it a sequence of numbers $\{x_n\}$ that under favorable circumstances converges to a zero of f. The recursion formula for the sequence is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

a. Show that the recursion formula for $f(x) = x^2 - a,\ a > 0$, can be written as $x_{n+1} = (x_n + a/x_n)/2$.

T **b.** Starting with $x_0 = 1$ and $a = 3$, calculate successive terms of the sequence until the display begins to repeat. What number is being approximated? Explain.

T **136. A recursive definition of $\pi/2$** If you start with $x_1 = 1$ and define the subsequent terms of $\{x_n\}$ by the rule $x_n = x_{n-1} + \cos x_{n-1}$, you generate a sequence that converges rapidly to $\pi/2$. **(a)** Try it. **(b)** Use the accompanying figure to explain why the convergence is so rapid.



COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the sequences in Exercises 137–148.

a. Calculate and then plot the first 25 terms of the sequence. Does the sequence appear to be bounded from above or below? Does it appear to converge or diverge? If it does converge, what is the limit L?

b. If the sequence converges, find an integer N such that $|a_n - L| \leq 0.01$ for $n \geq N$. How far in the sequence do you have to get for the terms to lie within 0.0001 of L?

137. $a_n = \sqrt[n]{n}$

138. $a_n = \left(1 + \frac{0.5}{n}\right)^n$

139. $a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{5^n}$

140. $a_1 = 1, \quad a_{n+1} = a_n + (-2)^n$

141. $a_n = \sin n$

142. $a_n = n \sin \frac{1}{n}$

143. $a_n = \frac{\sin n}{n}$

144. $a_n = \frac{\ln n}{n}$

145. $a_n = (0.9999)^n$

146. $a_n = (123456)^{1/n}$

147. $a_n = \frac{8^n}{n!}$

148. $a_n = \frac{n^{41}}{19^n}$

HIST
Blais
(1623

10.2 Infinite Series

An *infinite series* is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead we look at the result of summing just the first n terms of the sequence. The sum of the first n terms

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is an ordinary finite sum and can be calculated by normal addition. It is called the *nth partial sum*. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense that the terms of a sequence approach a limit, as discussed in Section 10.1.

For example, to assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

We saw this reindexing in starting a geometric series with the index $n = 0$ instead of the index $n = 1$, but we can use any other starting index value as well. We usually give preference to indexings that lead to simple expressions.

EXAMPLE 10 We can write the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \qquad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \qquad \text{or even} \qquad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}.$$

The partial sums remain the same no matter what indexing we choose to use. ■

Exercises 10.2

Finding *n*th Partial Sums

In Exercises 1–6, find a formula for the *n*th partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$

2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$

3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + \cdots$

4. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1}2^{n-1} + \cdots$

5. $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$

6. $\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

Series with Geometric Terms

In Exercises 7–14, write out the first eight terms of each series to show how the series starts. Then find the sum of the series or show that it diverges.

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$

8. $\sum_{n=2}^{\infty} \frac{1}{4^n}$

9. $\sum_{n=1}^{\infty} \left(1 - \frac{7}{4^n}\right)$

10. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

11. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$

12. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

13. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$

14. $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

In Exercises 15–18, determine if the geometric series converges or diverges. If a series converges, find its sum.

15. $1 + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \cdots$

16. $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$

17. $\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \left(\frac{1}{8}\right)^4 + \left(\frac{1}{8}\right)^5 + \cdots$

18. $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \left(\frac{-2}{3}\right)^5 + \left(\frac{-2}{3}\right)^6 + \cdots$

Repeating Decimals

Express each of the numbers in Exercises 19–26 as the ratio of two integers.

19. $0.\overline{23} = 0.23\,23\,23\ldots$

20. $0.\overline{234} = 0.234\,234\,234\ldots$

21. $0.\overline{7} = 0.7777\ldots$

22. $0.\overline{d} = 0.dddd\ldots$, where d is a digit

23. $0.0\overline{6} = 0.06666\ldots$

24. $1.\overline{414} = 1.414\,414\,414\ldots$

25. $1.24\overline{123} = 1.24\,123\,123\,123\ldots$

26. $3.\overline{142857} = 3.142857\,142857\ldots$

Using the *n*th-Term Test

In Exercises 27–34, use the *n*th-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

27. $\sum_{n=1}^{\infty} \frac{n}{n+10}$

28. $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$

29. $\sum_{n=0}^{\infty} \frac{1}{n+4}$

30. $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

31. $\sum_{n=1}^{\infty} \cos\frac{1}{n}$

32. $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$

33. $\sum_{n=1}^{\infty} \ln\frac{1}{n}$

34. $\sum_{n=0}^{\infty} \cos n\pi$

Telescoping Series

In Exercises 35–40, find a formula for the *n*th partial sum of the series and use it to determine if the series converges or diverges. If a series converges, find its sum.

35. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

36. $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$

37. $\sum_{n=1}^{\infty} \left(\ln\sqrt{n+1} - \ln\sqrt{n}\right)$

38. $\sum_{n=1}^{\infty} (\tan(n) - \tan(n-1))$

39. $\sum_{n=1}^{\infty} \left(\cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{n+2}\right)\right)$

40. $\sum_{n=1}^{\infty} \left(\sqrt{n+4} - \sqrt{n+3}\right)$

Find the sum of each series in Exercises 41–48.

41. $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

42. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

43. $\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$

44. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

45. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

46. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}}\right)$

47. $\sum_{n=1}^{\infty} \left(\frac{1}{\ln(n+2)} - \frac{1}{\ln(n+1)}\right)$

48. $\sum_{n=1}^{\infty} (\tan^{-1}(n) - \tan^{-1}(n+1))$

Convergence or Divergence

Which series in Exercises 49–68 converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

49. $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$

50. $\sum_{n=0}^{\infty} (\sqrt{2})^n$

51. $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{3}{2^n}$

52. $\sum_{n=1}^{\infty} (-1)^{n+1} n$

53. $\sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{2}\right)$

54. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$

55. $\sum_{n=0}^{\infty} e^{-2n}$

56. $\sum_{n=1}^{\infty} \ln\frac{1}{3^n}$

57. $\sum_{n=1}^{\infty} \frac{2}{10^n}$

58. $\sum_{n=0}^{\infty} \frac{1}{x^n}, \quad |x| > 1$

59. $\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$

60. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

61. $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$

62. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

63. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

64. $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$

65. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

66. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

67. $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

68. $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$

Geometric Series with a Variable *x*

In each of the geometric series in Exercises 69–72, write out the first few terms of the series to find a and r, and find the sum of the series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which the inequality holds and the series converges.

69. $\sum_{n=0}^{\infty} (-1)^n x^n$

70. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

71. $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

72. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n$

In Exercises 73–78, find the values of x for which the given geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

73. $\sum_{n=0}^{\infty} 2^n x^n$

74. $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$

75. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

76. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$

77. $\sum_{n=0}^{\infty} \sin^n x$

78. $\sum_{n=0}^{\infty} (\ln x)^n$

Theory and Examples

79. The series in Exercise 5 can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \quad \text{and} \quad \sum_{n=-1}^{\infty} \frac{1}{(n+3)(n+4)}.$$

Write it as a sum beginning with **(a)** $n = -2$, **(b)** $n = 0$, **(c)** $n = 5$.

80. The series in Exercise 6 can also be written as

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{5}{(n+1)(n+2)}.$$

Write it as a sum beginning with **(a)** $n = -1$, **(b)** $n = 3$, **(c)** $n = 20$.

81. Make up an infinite series of nonzero terms whose sum is

a. 1 **b.** -3 **c.** 0.

82. (*Continuation of Exercise 81.*) Can you make an infinite series of nonzero terms that converges to any number you want? Explain.

83. Show by example that $\sum(a_n/b_n)$ may diverge even though $\sum a_n$ and $\sum b_n$ converge and no b_n equals 0.

84. Find convergent geometric series $A = \sum a_n$ and $B = \sum b_n$ that illustrate the fact that $\sum a_n b_n$ may converge without being equal to AB.

85. Show by example that $\sum(a_n/b_n)$ may converge to something other than A/B even when $A = \sum a_n$, $B = \sum b_n \neq 0$, and no b_n equals 0.

86. If $\sum a_n$ converges and $a_n > 0$ for all n, can anything be said about $\sum(1/a_n)$? Give reasons for your answer.

87. What happens if you add a finite number of terms to a divergent series or delete a finite number of terms from a divergent series? Give reasons for your answer.

88. If $\sum a_n$ converges and $\sum b_n$ diverges, can anything be said about their term-by-term sum $\sum(a_n + b_n)$? Give reasons for your answer.

89. Make up a geometric series $\sum ar^{n-1}$ that converges to the number 5 if

a. $a = 2$ **b.** $a = 13/2$.

90. Find the value of b for which

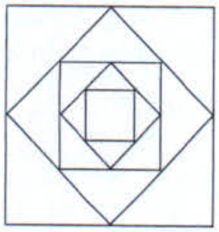
$$1 + e^b + e^{2b} + e^{3b} + \cdots = 9.$$

91. For what values of r does the infinite series

$$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + \cdots$$

converge? Find the sum of the series when it converges.

92. The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of 4 m^2. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



93. **Drug dosage** A patient takes a 300 mg tablet for the control of high blood pressure every morning at the same time. The concentration of the drug in the patient's system decays exponentially at a constant hourly rate of $k = 0.12$.

a. How many milligrams of the drug are in the patient's system just before the second tablet is taken? Just before the third tablet is taken?

b. In the long run, after taking the medication for at least six months, what quantity of drug is in the patient's body just before taking the next regularly scheduled morning tablet?

94. Show that the error $(L - s_n)$ obtained by replacing a convergent geometric series with one of its partial sums s_n is $ar^n/(1 - r)$.

95. **The Cantor set** To construct this set, we begin with the closed interval $[0, 1]$. From that interval, remove the middle open interval $(1/3, 2/3)$, leaving the two closed intervals $[0, 1/3]$ and $[2/3, 1]$. At the second step we remove the open middle third interval from each of those remaining. From $[0, 1/3]$ we remove the open interval $(1/9, 2/9)$, and from $[2/3, 1]$ we remove $(7/9, 8/9)$, leaving behind the four closed intervals $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$, and $[8/9, 1]$. At the next step, we remove the middle open third interval from each closed interval left behind, so $(1/27, 2/27)$ is removed from $[0, 1/9]$, leaving the closed intervals $[0, 1/27]$ and $[2/27, 1/9]$; $(7/27, 8/27)$ is removed from $[2/9, 1/3]$, leaving behind $[2/9, 7/27]$ and $[8/27, 1/3]$, and so forth. We continue this process repeatedly without stopping, at each step removing the open third interval from every closed interval remaining behind from the preceding step. The numbers remaining in the interval $[0, 1]$, after all open middle third intervals have been removed, are the points in the Cantor set (named after Georg Cantor, 1845–1918). The set has some interesting properties.

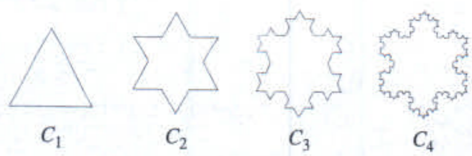
a. The Cantor set contains infinitely many numbers in $[0, 1]$. List 12 numbers that belong to the Cantor set.

b. Show, by summing an appropriate geometric series, that the total length of all the open middle third intervals that have been removed from $[0, 1]$ is equal to 1.

96. **Helga von Koch's snowflake curve** Helga von Koch's snowflake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

a. Find the length L_n of the nth curve C_n and show that $\lim_{n\to\infty} L_n = \infty$.

b. Find the area A_n of the region enclosed by C_n and show that $\lim_{n\to\infty} A_n = (8/5)\,A_1$.



10.3 The Integral Test

The most basic question we can ask about a series is whether it converges or not. In this section and the next two, we study this question, starting with series that have nonnegative terms. Such a series converges if its sequence of partial sums is bounded. If we establish that a given series does converge, we generally do not have a formula available for its sum. So for a convergent series, we need to investigate the error involved when using a partial sum to approximate its total sum.

Nondecreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$, so

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots.$$

Since the partial sums form a nondecreasing sequence, the Monotonic Sequence Theorem (Theorem 6, Section 10.1) gives the following result.

COROLLARY OF THEOREM 6 A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

The p-series for $p = 2$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493$$

The error in this approximation is then less than half the length of the interval, so the error is less than 0.005. Using a trigonometric *Fourier series* (studied in advanced calculus), it can be shown that S is equal to $\pi^2/6 \approx 1.64493$. ■

Exercises 10.3

Applying the Integral Test

Use the Integral Test to determine if the series in Exercises 1–10 converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$
3. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$
4. $\sum_{n=1}^{\infty} \frac{1}{n + 4}$
5. $\sum_{n=1}^{\infty} e^{-2n}$
6. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
7. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 4}$
8. $\sum_{n=2}^{\infty} \frac{\ln (n^2)}{n}$
9. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$
10. $\sum_{n=2}^{\infty} \frac{n - 4}{n^2 - 2n + 1}$

Determining Convergence or Divergence

Which of the series in Exercises 11–40 converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

11. $\sum_{n=1}^{\infty} \frac{1}{10^n}$
12. $\sum_{n=1}^{\infty} e^{-n}$
13. $\sum_{n=1}^{\infty} \frac{n}{n + 1}$
14. $\sum_{n=1}^{\infty} \frac{5}{n + 1}$
15. $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$
16. $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$
17. $\sum_{n=1}^{\infty} -\frac{1}{8^n}$
18. $\sum_{n=1}^{\infty} \frac{-8}{n}$
19. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$
20. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$
21. $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$
22. $\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3}$
23. $\sum_{n=0}^{\infty} \frac{-2}{n + 1}$
24. $\sum_{n=1}^{\infty} \frac{1}{2n - 1}$
25. $\sum_{n=1}^{\infty} \frac{2^n}{n + 1}$
26. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$
27. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$
28. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n} + 1)}$
29. $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$
30. $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$
31. $\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$
32. $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln^2 n)}$
33. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$
34. $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$
35. $\sum_{n=1}^{\infty} \frac{e^n}{1 + e^{2n}}$
36. $\sum_{n=1}^{\infty} \frac{2}{1 + e^n}$
37. $\sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1 + n^2}$
38. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$
39. $\sum_{n=1}^{\infty} \operatorname{sech} n$
40. $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$

Theory and Examples

For what values of a, if any, do the series in Exercises 41 and 42 converge?

41. $\sum_{n=1}^{\infty} \left(\frac{a}{n + 2} - \frac{1}{n + 4}\right)$
42. $\sum_{n=3}^{\infty} \left(\frac{1}{n - 1} - \frac{2a}{n + 1}\right)$

43. **a.** Draw illustrations like those in Figures 10.11a and 10.11b to show that the partial sums of the harmonic series satisfy the inequalities

$$\ln(n + 1) = \int_1^{n+1} \frac{1}{x}dx \le 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$
$$\le 1 + \int_1^n \frac{1}{x}dx = 1 + \ln n.$$

T **b.** There is absolutely no empirical evidence for the divergence of the harmonic series even though we know it diverges. The partial sums just grow too slowly. To see what we mean, suppose you had started with $s_1 = 1$ the day the universe was formed, 13 billion years ago, and added a new term every *second*. About how large would the partial sum s_n be today, assuming a 365-day year?

44. Are there any values of x for which $\sum_{n=1}^{\infty}(1/nx)$ converges? Give reasons for your answer.

45. Is it true that if $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then there is also a divergent series $\sum_{n=1}^{\infty} b_n$ of positive numbers with $b_n < a_n$ for every n? Is there a "smallest" divergent series of positive numbers? Give reasons for your answers.

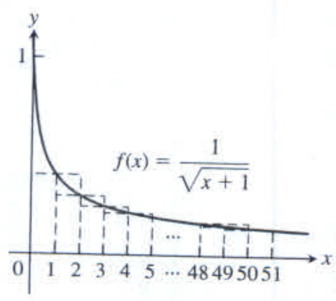
46. (*Continuation of Exercise 45.*) Is there a "largest" convergent series of positive numbers? Explain.

47. **$\sum_{n=1}^{\infty} (1/\sqrt{n + 1})$ diverges**

a. Use the accompanying graph to show that the partial sum $s_{50} = \sum_{n=1}^{50} (1/\sqrt{n + 1})$ satisfies

$$\int_1^{51} \frac{1}{\sqrt{x + 1}}\,dx < s_{50} < \int_0^{50} \frac{1}{\sqrt{x + 1}}\,dx.$$

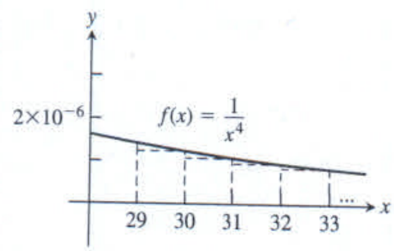
Conclude that $11.5 < s_{50} < 12.3$.



b. What should n be in order that the partial sum $s_n = \sum_{i=1}^{n} \left(1/\sqrt{i+1}\right)$ satisfy $s_n > 1000$?

48. $\sum_{n=1}^{\infty} (1/n^4)$ **converges**

a. Use the accompanying graph to find an upper bound for the error if $s_{30} = \sum_{n=1}^{30} (1/n^4)$ is used to estimate the value of $\sum_{n=1}^{\infty} (1/n^4)$.



b. Find n so that the partial sum $s_n = \sum_{i=1}^{n} (1/i^4)$ estimates the value of $\sum_{n=1}^{\infty} (1/n^4)$ with an error of at most 0.000001.

49. Estimate the value of $\sum_{n=1}^{\infty} (1/n^3)$ to within 0.01 of its exact value.

50. Estimate the value of $\sum_{n=2}^{\infty} (1/(n^2+4))$ to within 0.1 of its exact value.

51. How many terms of the convergent series $\sum_{n=1}^{\infty} (1/n^{1.1})$ should be used to estimate its value with error at most 0.00001?

52. How many terms of the convergent series $\sum_{n=4}^{\infty} (1/n(\ln n)^3)$ should be used to estimate its value with error at most 0.01?

53. **The Cauchy condensation test** The Cauchy condensation test says: Let $\{a_n\}$ be a nonincreasing sequence ($a_n \geq a_{n+1}$ for all n) of positive terms that converges to 0. Then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. For example, $\sum(1/n)$ diverges because $\sum 2^n \cdot (1/2^n) = \sum 1$ diverges. Show why the test works.

54. Use the Cauchy condensation test from Exercise 53 to show that

a. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges;

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

55. **Logarithmic p-series**

a. Show that the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} \quad (p \text{ a positive constant})$$

converges if and only if $p > 1$.

b. What implications does the fact in part (a) have for the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}?$$

Give reasons for your answer.

56. (*Continuation of Exercise 55.*) Use the result in Exercise 55 to determine which of the following series converge and which diverge. Support your answer in each case.

a. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$

b. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$

c. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln(n^3)}$

d. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

57. **Euler's constant** Graphs like those in Figure 10.11 suggest that as n increases there is little change in the difference between the sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and the integral

$$\ln n = \int_1^n \frac{1}{x}\,dx.$$

To explore this idea, carry out the following steps.

a. By taking $f(x) = 1/x$ in the proof of Theorem 9, show that

$$\ln(n+1) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \ln n$$

or

$$0 < \ln(n+1) - \ln n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \leq 1.$$

Thus, the sequence

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

is bounded from below and from above.

b. Show that

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx = \ln(n+1) - \ln n,$$

and use this result to show that the sequence $\{a_n\}$ in part (a) is decreasing.

Since a decreasing sequence that is bounded from below converges, the numbers a_n defined in part (a) converge:

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \to \gamma.$$

The number γ, whose value is 0.5772 . . ., is called *Euler's constant*.

58. Use the Integral Test to show that the series

$$\sum_{n=0}^{\infty} e^{-n^2}$$

converges.

59. a. For the series $\sum(1/n^3)$, use the inequalities in Equation (2) with $n = 10$ to find an interval containing the sum S.

b. As in Example 5, use the midpoint of the interval found in part (a) to approximate the sum of the series. What is the maximum error for your approximation?

60. Repeat Exercise 59 using the series $\sum(1/n^4)$.

61. **Area** Consider the sequence $\{1/n\}_{n=1}^{\infty}$. On each subinterval $(1/(n+1), 1/n)$ within the interval $[0, 1]$, erect the rectangle with area a_n having height $1/n$ and width equal to the length of the subinterval. Find the total area $\sum a_n$ of all the rectangles. (*Hint:* Use the result of Example 5 in Section 10.2.)

62. **Area** Repeat Exercise 61, using trapezoids instead of rectangles. That is, on the subinterval $(1/(n+1), 1/n)$, let a_n denote the area of the trapezoid having heights $y = 1/(n+1)$ at $x = 1/(n+1)$ and $y = 1/n$ at $x = 1/n$.

for n sufficiently large. Then taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\ln n}{n^{1/4}}$$

$$= \lim_{n\to\infty}\frac{1/n}{(1/4)n^{-3/4}} \qquad \text{l'Hôpital's Rule}$$

$$= \lim_{n\to\infty}\frac{4}{n^{1/4}} = 0.$$

Since $\sum b_n = \sum(1/n^{5/4})$ is a p-series with $p > 1$, it converges, so $\sum a_n$ converges by Part 2 of the Limit Comparison Test. ∎

Exercises 10.4

Comparison Test

In Exercises 1–8, use the Comparison Test to determine if each series converges or diverges.

1. $\sum_{n=1}^{\infty}\frac{1}{n^2+30}$
2. $\sum_{n=1}^{\infty}\frac{n-1}{n^4+2}$
3. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$
4. $\sum_{n=2}^{\infty}\frac{n+2}{n^2-n}$
5. $\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^{3/2}}$
6. $\sum_{n=1}^{\infty}\frac{1}{n3^n}$
7. $\sum_{n=1}^{\infty}\sqrt{\frac{n+4}{n^4+4}}$
8. $\sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

Limit Comparison Test

In Exercises 9–16, use the Limit Comparison Test to determine if each series converges or diverges.

9. $\sum_{n=1}^{\infty}\frac{n-2}{n^3-n^2+3}$
(*Hint:* Limit Comparison with $\sum_{n=1}^{\infty}(1/n^2)$)
10. $\sum_{n=1}^{\infty}\sqrt{\frac{n+1}{n^2+2}}$
(*Hint:* Limit Comparison with $\sum_{n=1}^{\infty}(1/\sqrt{n})$)
11. $\sum_{n=2}^{\infty}\frac{n(n+1)}{(n^2+1)(n-1)}$
12. $\sum_{n=1}^{\infty}\frac{2^n}{3+4^n}$
13. $\sum_{n=1}^{\infty}\frac{5^n}{\sqrt{n}\,4^n}$
14. $\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^n$
15. $\sum_{n=2}^{\infty}\frac{1}{\ln n}$
(*Hint:* Limit Comparison with $\sum_{n=2}^{\infty}(1/n)$)
16. $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2}\right)$
(*Hint:* Limit Comparison with $\sum_{n=1}^{\infty}(1/n^2)$)

Determining Convergence or Divergence

Which of the series in Exercises 17–54 converge, and which diverge? Use any method, and give reasons for your answers.

17. $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}+\sqrt[3]{n}}$
18. $\sum_{n=1}^{\infty}\frac{3}{n+\sqrt{n}}$
19. $\sum_{n=1}^{\infty}\frac{\sin^2 n}{2^n}$
20. $\sum_{n=1}^{\infty}\frac{1+\cos n}{n^2}$
21. $\sum_{n=1}^{\infty}\frac{2n}{3n-1}$
22. $\sum_{n=1}^{\infty}\frac{n+1}{n^2\sqrt{n}}$
23. $\sum_{n=1}^{\infty}\frac{10n+1}{n(n+1)(n+2)}$
24. $\sum_{n=3}^{\infty}\frac{5n^3-3n}{n^2(n-2)(n^2+5)}$
25. $\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$
26. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+2}}$
27. $\sum_{n=3}^{\infty}\frac{1}{\ln(\ln n)}$
28. $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^3}$
29. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\ln n}$
30. $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{3/2}}$
31. $\sum_{n=1}^{\infty}\frac{1}{1+\ln n}$
32. $\sum_{n=2}^{\infty}\frac{\ln(n+1)}{n+1}$
33. $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-1}}$
34. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$
35. $\sum_{n=1}^{\infty}\frac{1-n}{n2^n}$
36. $\sum_{n=1}^{\infty}\frac{n+2^n}{n^2 2^n}$
37. $\sum_{n=1}^{\infty}\frac{1}{3^{n-1}+1}$
38. $\sum_{n=1}^{\infty}\frac{3^{n-1}+1}{3^n}$
39. $\sum_{n=1}^{\infty}\frac{n+1}{n^2+3n}\cdot\frac{1}{5n}$
40. $\sum_{n=1}^{\infty}\frac{2^n+3^n}{3^n+4^n}$
41. $\sum_{n=1}^{\infty}\frac{2^n-n}{n2^n}$
42. $\sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{n}\,e^n}$
43. $\sum_{n=2}^{\infty}\frac{1}{n!}$
(*Hint:* First show that $(1/n!) \le (1/n(n-1))$ for $n \ge 2$.)
44. $\sum_{n=1}^{\infty}\frac{(n-1)!}{(n+2)!}$
45. $\sum_{n=1}^{\infty}\sin\frac{1}{n}$
46. $\sum_{n=1}^{\infty}\tan\frac{1}{n}$
47. $\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{n^{1.1}}$
48. $\sum_{n=1}^{\infty}\frac{\sec^{-1}n}{n^{1.3}}$
49. $\sum_{n=1}^{\infty}\frac{\coth n}{n^2}$
50. $\sum_{n=1}^{\infty}\frac{\tanh n}{n^2}$
51. $\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}$
52. $\sum_{n=1}^{\infty}\frac{\sqrt[n]{n}}{n^2}$
53. $\sum_{n=1}^{\infty}\frac{1}{1+2+3+\cdots+n}$
54. $\sum_{n=1}^{\infty}\frac{1}{1+2^2+3^2+\cdots+n^2}$

Theory and Examples

55. Prove **(a)** Part 2 and **(b)** Part 3 of the Limit Comparison Test.
56. If $\sum_{n=1}^{\infty}a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty}(a_n/n)$? Explain.
57. Suppose that $a_n > 0$ and $b_n > 0$ for $n \ge N$ (N an integer). If $\lim_{n\to\infty}(a_n/b_n) = \infty$ and $\sum a_n$ converges, can anything be said about $\sum b_n$? Give reasons for your answer.
58. Prove that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n{}^2$ converges.

59. Suppose that $a_n > 0$ and $\lim_{n\to\infty} a_n = \infty$. Prove that $\sum a_n$ diverges.

60. Suppose that $a_n > 0$ and $\lim_{n\to\infty} n^2 a_n = 0$. Prove that $\sum a_n$ converges.

61. Show that $\sum_{n=2}^{\infty} ((\ln n)^q / n^p)$ converges for $-\infty < q < \infty$ and $p > 1$.

(*Hint:* Limit Comparison with $\sum_{n=2}^{\infty} 1/n^r$ for $1 < r < p$.)

62. (*Continuation of Exercise 61.*) Show that $\sum_{n=2}^{\infty} ((\ln n)^q / n^p)$ diverges for $-\infty < q < \infty$ and $0 < p < 1$.

(*Hint:* Limit Comparison with an appropriate p-series.)

63. **Decimal numbers** Any real number in the interval $[0, 1]$ can be represented by a decimal (not necessarily unique) as

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots,$$

where d_i is one of the integers $0, 1, 2, 3, \ldots, 9$. Prove that the series on the right-hand side always converges.

64. If $\sum a_n$ is a convergent series of positive terms, prove that $\sum \sin(a_n)$ converges.

In Exercises 65–70, use the results of Exercises 61 and 62 to determine if each series converges or diverges.

65. $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^3}{n^4}$

66. $\displaystyle\sum_{n=2}^{\infty} \sqrt{\frac{\ln n}{n}}$

67. $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^{1000}}{n^{1.001}}$

68. $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^{1/5}}{n^{0.99}}$

69. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.1}(\ln n)^3}$

70. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n \cdot \ln n}}$

COMPUTER EXPLORATIONS

71. It is not yet known whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$$

converges or diverges. Use a CAS to explore the behavior of the series by performing the following steps.

a. Define the sequence of partial sums

$$s_k = \sum_{n=1}^{k} \frac{1}{n^3 \sin^2 n}.$$

What happens when you try to find the limit of s_k as $k \to \infty$? Does your CAS find a closed form answer for this limit?

b. Plot the first 100 points (k, s_k) for the sequence of partial sums. Do they appear to converge? What would you estimate the limit to be?

c. Next plot the first 200 points (k, s_k). Discuss the behavior in your own words.

d. Plot the first 400 points (k, s_k). What happens when $k = 355$? Calculate the number $355/113$. Explain from you calculation what happened at $k = 355$. For what values of k would you guess this behavior might occur again?

72. a. Use Theorem 8 to show that

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)} \right)$$

where $S = \sum_{n=1}^{\infty} (1/n^2)$, the sum of a convergent p-series.

b. From Example 5, Section 10.2, show that

$$S = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}.$$

c. Explain why taking the first M terms in the series in part (b) gives a better approximation to S than taking the first M terms in the original series $\sum_{n=1}^{\infty} (1/n^2)$.

d. We know the exact value of S is $\pi^2/6$. Which of the sums

$$\sum_{n=1}^{1000000} \frac{1}{n^2} \quad \text{or} \quad 1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$$

gives a better approximation to S?

10.5 Absolute Convergence; The Ratio and Root Tests

When some of the terms of a series are positive and others are negative, the series may or may not converge. For example, the geometric series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5\left(\frac{-1}{4}\right)^n \tag{1}$$

converges (since $|r| = \frac{1}{4} < 1$), whereas the different geometric series

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots = \sum_{n=0}^{\infty} \left(\frac{-5}{4}\right)^n \tag{2}$$

diverges (since $|r| = 5/4 > 1$). In series (1), there is some cancelation in the partial sums, which may be assisting the convergence property of the series. However, if we make all of the terms positive in series (1) to form the new series

Solution We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even}. \end{cases}$$

Therefore,

$$\frac{1}{2} \le \sqrt[n]{|a_n|} \le \frac{\sqrt[n]{n}}{2}.$$

Since $\sqrt[n]{n} \to 1$ (Section 10.1, Theorem 5), we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges absolutely by the Root Test. ■

EXAMPLE 4 Which of the following series converge, and which diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution We apply the Root Test to each series, noting that each series has positive terms.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\displaystyle\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1^2}{2} < 1.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges because $\displaystyle\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\left(\sqrt[n]{n}\right)^3} \to \frac{2}{1^3} > 1.$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\displaystyle\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$ ■

Exercises 10.5

Using the Ratio Test

In Exercises 1–8, use the Ratio Test to determine if each series converges absolutely or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{(-4)^n}$
6. $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$
7. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2(n+2)!}{n!\, 3^{2n}}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

Using the Root Test

In Exercises 9–16, use the Root Test to determine if each series converges absolutely or diverges.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$
11. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$
12. $\displaystyle\sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{-8}{(3 + (1/n))^{2n}}$
14. $\displaystyle\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$
15. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\left(1 - \frac{1}{n}\right)^{n^2}$

    (*Hint:* $\lim_{n\to\infty} (1 + x/n)^n = e^x$)
16. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1+n}}$

Determining Convergence or Divergence

In Exercises 17–44, use any method to determine if the series converges or diverges. Give reasons for your answer.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$
18. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n}$
19. $\displaystyle\sum_{n=1}^{\infty} n!(-e)^{-n}$
20. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$
21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$
22. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$

23. $\sum_{n=1}^{\infty} \frac{2 + (-1)^n}{1.25^n}$

24. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$

25. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{3}{n}\right)^n$

26. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{3n}\right)^n$

27. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

28. $\sum_{n=1}^{\infty} \frac{(-\ln n)^n}{n^n}$

29. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$

30. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

31. $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$

32. $\sum_{n=1}^{\infty} \frac{n \ln n}{(-2)^n}$

33. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{n!}$

34. $\sum_{n=1}^{\infty} e^{-n}(n^3)$

35. $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$

36. $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$

37. $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

38. $\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$

39. $\sum_{n=2}^{\infty} \frac{-n}{(\ln n)^n}$

40. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{(n/2)}}$

41. $\sum_{n=1}^{\infty} \frac{n! \ln n}{n(n+2)!}$

42. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 2^n}$

43. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

44. $\sum_{n=1}^{\infty} \frac{(2n+3)(2^n+3)}{3^n+2}$

Recursively Defined Terms Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 45–54 converge, and which diverge? Give reasons for your answers.

45. $a_1 = 2, \quad a_{n+1} = \frac{1 + \sin n}{n} a_n$

46. $a_1 = 1, \quad a_{n+1} = \frac{1 + \tan^{-1} n}{n} a_n$

47. $a_1 = \frac{1}{3}, \quad a_{n+1} = \frac{3n - 1}{2n + 5} a_n$

48. $a_1 = 3, \quad a_{n+1} = \frac{n}{n+1} a_n$

49. $a_1 = 2, \quad a_{n+1} = \frac{2}{n} a_n$

50. $a_1 = 5, \quad a_{n+1} = \frac{\sqrt[n]{n}}{2} a_n$

51. $a_1 = 1, \quad a_{n+1} = \frac{1 + \ln n}{n} a_n$

52. $a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{n + \ln n}{n + 10} a_n$

53. $a_1 = \frac{1}{3}, \quad a_{n+1} = \sqrt[n]{a_n}$

54. $a_1 = \frac{1}{2}, \quad a_{n+1} = (a_n)^{n+1}$

Convergence or Divergence

Which of the series in Exercises 55–62 converge, and which diverge? Give reasons for your answers.

55. $\sum_{n=1}^{\infty} \frac{2^n n! n!}{(2n)!}$

56. $\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!}{n!(n+1)!(n+2)!}$

57. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

58. $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^n}{n^{(n^2)}}$

59. $\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$

60. $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

61. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{4^n 2^n n!}$

62. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{[2 \cdot 4 \cdot \cdots \cdot (2n)](3^n + 1)}$

Theory and Examples

63. Neither the Ratio Test nor the Root Test helps with p-series. Try them on

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and show that both tests fail to provide information about convergence.

64. Show that neither the Ratio Test nor the Root Test provides information about the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p} \qquad (p \text{ constant}).$$

65. Let $a_n = \begin{cases} n/2^n, & \text{if } n \text{ is a prime number} \\ 1/2^n, & \text{otherwise.} \end{cases}$

Does $\sum a_n$ converge? Give reasons for your answer.

66. Show that $\sum_{n=1}^{\infty} 2^{(n^2)}/n!$ diverges. Recall from the Laws of Exponents that $2^{(n^2)} = (2^n)^n$.

10.6 Alternating Series and Conditional Convergence

A series in which the terms are alternately positive and negative is an **alternating series**. Here are three examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \qquad (1)$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \qquad (2)$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots \qquad (3)$$

Exercises 10.6

Determining Convergence or Divergence

In Exercises 1–14, determine if the alternating series converges or diverges. Some of the series do not satisfy the conditions of the Alternating Series Test.

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$

3. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n3^n}$

4. $\sum_{n=2}^{\infty} (-1)^{n} \frac{4}{(\ln n)^2}$

5. $\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n^2+1}$

6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4}$

7. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n^2}$

8. $\sum_{n=1}^{\infty} (-1)^{n} \frac{10^n}{(n+1)!}$

9. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^n$

10. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$

11. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

12. $\sum_{n=1}^{\infty} (-1)^{n} \ln\left(1+\frac{1}{n}\right)$

13. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

14. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n+1}}{\sqrt{n}+1}$

Absolute and Conditional Convergence

Which of the series in Exercises 15–48 converge absolutely, which converge, and which diverge? Give reasons for your answers.

15. $\sum_{n=1}^{\infty} (-1)^{n+1}(0.1)^n$

16. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}$

17. $\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{\sqrt{n}}$

18. $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$

19. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$

20. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$

21. $\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n+3}$

22. $\sum_{n=1}^{\infty} (-1)^{n} \frac{\sin n}{n^2}$

23. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3+n}{5+n}$

24. $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$

25. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$

26. $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\sqrt[n]{10}\right)$

27. $\sum_{n=1}^{\infty} (-1)^{n} n^2 (2/3)^n$

28. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$

29. $\sum_{n=1}^{\infty} (-1)^{n} \frac{\tan^{-1} n}{n^2+1}$

30. $\sum_{n=1}^{\infty} (-1)^{n} \frac{\ln n}{n - \ln n}$

31. $\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$

32. $\sum_{n=1}^{\infty} (-5)^{-n}$

33. $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$

34. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$

35. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n\sqrt{n}}$

36. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$

37. $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^n}{(2n)^n}$

38. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$

39. $\sum_{n=1}^{\infty} (-1)^{n} \frac{(2n)!}{2^n n! n}$

40. $\sum_{n=1}^{\infty} (-1)^{n} \frac{(n!)^2 3^n}{(2n+1)!}$

41. $\sum_{n=1}^{\infty} (-1)^{n}\left(\sqrt{n+1}-\sqrt{n}\right)$

42. $\sum_{n=1}^{\infty} (-1)^{n}\left(\sqrt{n^2+n}-n\right)$

43. $\sum_{n=1}^{\infty} (-1)^{n}\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$

44. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$

45. $\sum_{n=1}^{\infty} (-1)^{n} \operatorname{sech} n$

46. $\sum_{n=1}^{\infty} (-1)^{n} \operatorname{csch} n$

47. $\frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} + \frac{1}{12} - \frac{1}{14} + \cdots$

48. $1 + \frac{1}{4} - \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} + \cdots$

Error Estimation

In Exercises 49–52, estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

49. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

50. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$

51. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.01)^n}{n}$ As you will see in Section 10.7, the sum is $\ln(1.01)$.

52. $\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n, \quad 0 < t < 1$

In Exercises 53–56, determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001.

53. $\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n^2+3}$

54. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$

55. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\left(n+3\sqrt{n}\right)^3}$

56. $\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{\ln(\ln(n+2))}$

T Approximate the sums in Exercises 57 and 58 with an error of magnitude less than 5×10^{-6}.

57. $\sum_{n=0}^{\infty} (-1)^{n} \frac{1}{(2n)!}$ As you will see in Section 10.9, the sum is cos 1, the cosine of 1 radian.

58. $\sum_{n=0}^{\infty} (-1)^{n} \frac{1}{n!}$ As you will see in Section 10.9, the sum is e^{-1}.

Theory and Examples

59. **a.** The series

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \cdots + \frac{1}{3^n} - \frac{1}{2^n} + \cdots$$

does not meet one of the conditions of Theorem 14. Which one?

b. Use Theorem 17 to find the sum of the series in part (a).

T **60.** The limit L of an alternating series that satisfies the conditions of Theorem 15 lies between the values of any two consecutive partial sums. This suggests using the average

$$\frac{s_n + s_{n+1}}{2} = s_n + \frac{1}{2}(-1)^{n+2}a_{n+1}$$

to estimate L. Compute

$$s_{20} + \frac{1}{2}\cdot\frac{1}{21}$$

as an approximation to the sum of the alternating harmonic series. The exact sum is $\ln 2 = 0.69314718\ldots$.

61. The sign of the remainder of an alternating series that satisfies the conditions of Theorem 15 Prove the assertion in Theorem 16 that whenever an alternating series satisfying the conditions of Theorem 15 is approximated with one of its partial sums, then the remainder (sum of the unused terms) has the same sign as the first unused term. (*Hint:* Group the remainder's terms in consecutive pairs.)

62. Show that the sum of the first $2n$ terms of the series

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots$$

is the same as the sum of the first n terms of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \cdots.$$

Do these series converge? What is the sum of the first $2n + 1$ terms of the first series? If the series converge, what is their sum?

63. Show that if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

64. Show that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|.$$

65. Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge absolutely, then so do the following.

a. $\sum_{n=1}^{\infty}(a_n + b_n)$ **b.** $\sum_{n=1}^{\infty}(a_n - b_n)$

c. $\sum_{n=1}^{\infty} ka_n$ (k any number)

66. Show by example that $\sum_{n=1}^{\infty} a_n b_n$ may diverge even if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

67. If $\sum a_n$ converges absolutely, prove that $\sum a_n^2$ converges.

68. Does the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n^2}\right)$$

converge or diverge? Justify your answer.

T **69.** In the alternating harmonic series, suppose the goal is to arrange the terms to get a new series that converges to $-1/2$. Start the new arrangement with the first negative term, which is $-1/2$. Whenever you have a sum that is less than or equal to $-1/2$, start introducing positive terms, taken in order, until the new total is greater than $-1/2$. Then add negative terms until the total is less than or equal to $-1/2$ again. Continue this process until your partial sums have been above the target at least three times and finish at or below it. If s_n is the sum of the first n terms of your new series, plot the points (n, s_n) to illustrate how the sums are behaving.

70. Outline of the proof of the Rearrangement Theorem (Theorem 17)

a. Let ϵ be a positive real number, let $L = \sum_{n=1}^{\infty} a_n$, and let $s_k = \sum_{n=1}^{k} a_n$. Show that for some index N_1 and for some index $N_2 \ge N_1$,

$$\sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2} \quad \text{and} \quad |s_{N_2} - L| < \frac{\epsilon}{2}.$$

Since all the terms $a_1, a_2, \ldots, a_{N_2}$ appear somewhere in the sequence $\{b_n\}$, there is an index $N_3 \ge N_2$ such that if $n \ge N_3$, then $\left(\sum_{k=1}^{n} b_k\right) - s_{N_2}$ is at most a sum of terms a_m with $m \ge N_1$. Therefore, if $n \ge N_3$,

$$\left|\sum_{k=1}^{n} b_k - L\right| \le \left|\sum_{k=1}^{n} b_k - s_{N_2}\right| + |s_{N_2} - L|$$

$$\le \sum_{k=N_1}^{\infty} |a_k| + |s_{N_2} - L| < \epsilon.$$

b. The argument in part (a) shows that if $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$. Now show that because $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} |b_n|$ converges to $\sum_{n=1}^{\infty} |a_n|$.

10.7 Power Series

Now that we can test many infinite series of numbers for convergence, we can study sums that look like "infinite polynomials." We call these sums *power series* because they are defined as infinite series of powers of some variable, in our case x. Like polynomials, power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series. With power series we can extend the methods of calculus we have developed to a vast array of functions, making the techniques of calculus applicable in a much wider setting.