Exercises 14.1

Domain, Range, and Level Curves

In Exercises 1–4, find the specific function values.

1. $f(x, y) = x^2 + xy^3$
   a. $f(0, 0)$ b. $f(-1, 1)$
   c. $f(2, 3)$ d. $f(-3, -2)$

2. $f(x, y) = \sin(xy)$
   a. $f\left(2, \frac{\pi}{6}\right)$ b. $f\left(-3, \frac{\pi}{12}\right)$
   c. $f\left(\pi, \frac{1}{4}\right)$ d. $f\left(-\frac{\pi}{2}, -7\right)$

3. $f(x, y, z) = \dfrac{x - y}{y^2 + z^2}$
   a. $f(3, -1, 2)$ b. $f\left(1, \frac{1}{2}, -\frac{1}{4}\right)$
   c. $f\left(0, -\frac{1}{3}, 0\right)$ d. $f(2, 2, 100)$

4. $f(x, y, z) = \sqrt{49 - x^2 - y^2 - z^2}$
   a. $f(0, 0, 0)$ b. $f(2, -3, 6)$
   c. $f(-1, 2, 3)$ d. $f\left(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}\right)$

In Exercises 5–12, find and sketch the domain for each function.

5. $f(x, y) = \sqrt{y - x - 2}$
6. $f(x, y) = \ln(x^2 + y^2 - 4)$
7. $f(x, y) = \dfrac{(x - 1)(y + 2)}{(y - x)(y - x^3)}$
8. $f(x, y) = \dfrac{\sin(xy)}{x^2 + y^2 - 25}$
9. $f(x, y) = \cos^{-1}(y - x^2)$
10. $f(x, y) = \ln(xy + x - y - 1)$
11. $f(x, y) = \sqrt{(x^2 - 4)(y^2 - 9)}$
12. $f(x, y) = \dfrac{1}{\ln(4 - x^2 - y^2)}$

In Exercises 13–16, find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c. We refer to these level curves as a contour map.

13. $f(x, y) = x + y - 1, \quad c = -3, -2, -1, 0, 1, 2, 3$
14. $f(x, y) = x^2 + y^2, \quad c = 0, 1, 4, 9, 16, 25$
15. $f(x, y) = xy, \quad c = -9, -4, -1, 0, 1, 4, 9$
16. $f(x, y) = \sqrt{25 - x^2 - y^2}, \quad c = 0, 1, 2, 3, 4$

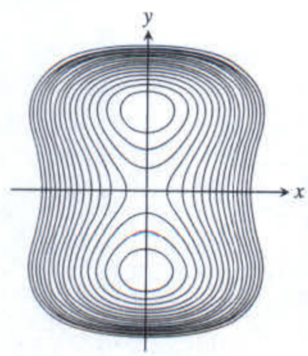
In Exercises 17–30, **(a)** find the function's domain, **(b)** find the function's range, **(c)** describe the function's level curves, **(d)** find the boundary of the function's domain, **(e)** determine if the domain is an open region, a closed region, or neither, and **(f)** decide if the domain is bounded or unbounded.

17. $f(x, y) = y - x$ **18.** $f(x, y) = \sqrt{y - x}$
19. $f(x, y) = 4x^2 + 9y^2$ **20.** $f(x, y) = x^2 - y^2$
21. $f(x, y) = xy$ **22.** $f(x, y) = y/x^2$
23. $f(x, y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$ **24.** $f(x, y) = \sqrt{9 - x^2 - y^2}$
25. $f(x, y) = \ln(x^2 + y^2)$ **26.** $f(x, y) = e^{-(x^2+y^2)}$
27. $f(x, y) = \sin^{-1}(y - x)$ **28.** $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$
29. $f(x, y) = \ln(x^2 + y^2 - 1)$ **30.** $f(x, y) = \ln(9 - x^2 - y^2)$

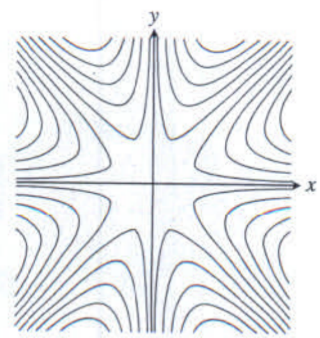
Matching Surfaces with Level Curves

Exercises 31–36 show level curves for the functions graphed in (a)–(f) on the following page. Match each set of curves with the appropriate function.

31.



32.



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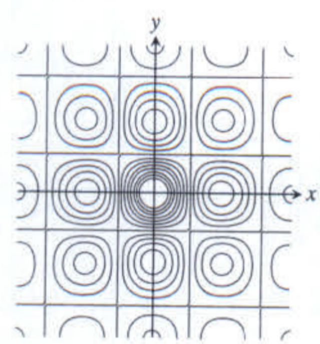
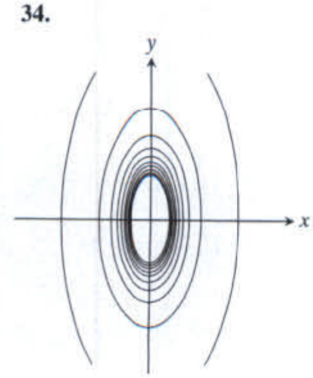
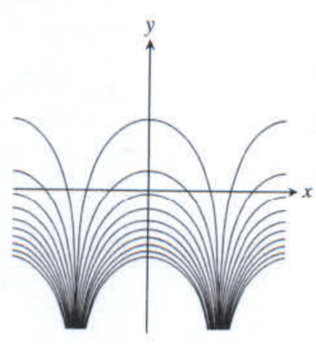


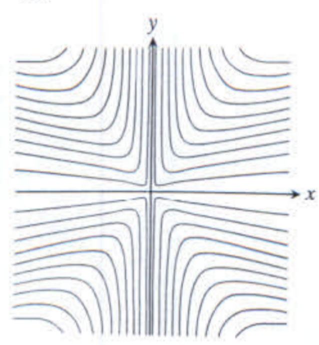
34.



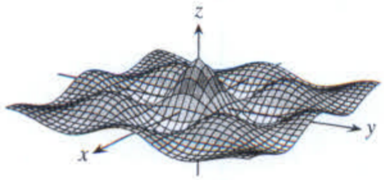
35.



36.

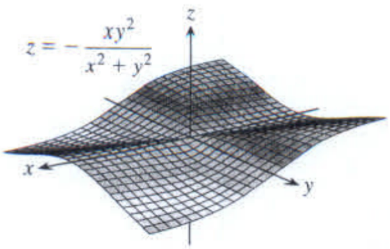


a.



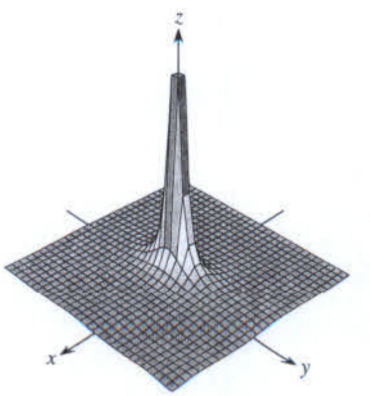
$z = (\cos x)(\cos y)\, e^{-\sqrt{x^2 + y^2}/4}$

b.



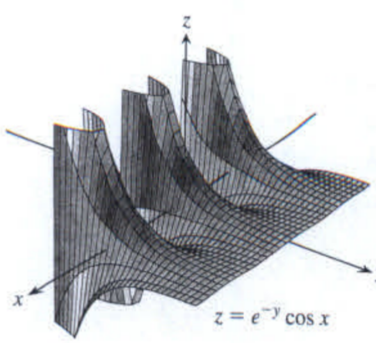
$z = -\dfrac{xy^2}{x^2 + y^2}$

c.



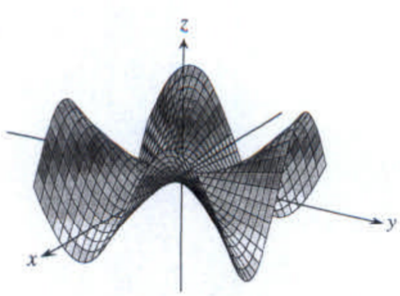
$z = \dfrac{1}{4x^2 + y^2}$

d.



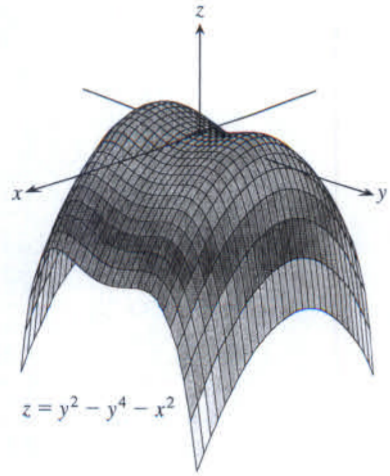
$z = e^{-y} \cos x$

e.



$z = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$

f.



$z = y^2 - y^4 - x^2$

Functions of Two Variables

Display the values of the functions in Exercises 37–48 in two ways: (a) by sketching the surface $z = f(x, y)$ and (b) by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.

37. $f(x, y) = y^2$

38. $f(x, y) = \sqrt{x}$

39. $f(x, y) = x^2 + y^2$

40. $f(x, y) = \sqrt{x^2 + y^2}$

41. $f(x, y) = x^2 - y$

42. $f(x, y) = 4 - x^2 - y^2$

43. $f(x, y) = 4x^2 + y^2$

44. $f(x, y) = 6 - 2x - 3y$

45. $f(x, y) = 1 - |y|$

46. $f(x, y) = 1 - |x| - |y|$

47. $f(x, y) = \sqrt{x^2 + y^2 + 4}$

48. $f(x, y) = \sqrt{x^2 + y^2 - 4}$

Finding Level Curves

In Exercises 49–52, find an equation for and sketch the graph of the level curve of the function $f(x, y)$ that passes through the given point.

49. $f(x, y) = 16 - x^2 - y^2, \quad (2\sqrt{2}, \sqrt{2})$

50. $f(x, y) = \sqrt{x^2 - 1}, \quad (1, 0)$

51. $f(x, y) = \sqrt{x + y^2 - 3}, \quad (3, -1)$

52. $f(x, y) = \dfrac{2y - x}{x + y + 1}, \quad (-1, 1)$

Sketching Level Surfaces

In Exercises 53–60, sketch a typical level surface for the function.

53. $f(x, y, z) = x^2 + y^2 + z^2$

54. $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

55. $f(x, y, z) = x + z$

56. $f(x, y, z) = z$

57. $f(x, y, z) = x^2 + y^2$

58. $f(x, y, z) = y^2 + z^2$

59. $f(x, y, z) = z - x^2 - y^2$

60. $f(x, y, z) = (x^2/25) + (y^2/16) + (z^2/9)$

Finding Level Surfaces

In Exercises 61–64, find an equation for the level surface of the function through the given point.

61. $f(x, y, z) = \sqrt{x - y} - \ln z, \quad (3, -1, 1)$

62. $f(x, y, z) = \ln(x^2 + y + z^2), \quad (-1, 2, 1)$

63. $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $\left(1, -1, \sqrt{2}\right)$

64. $g(x, y, z) = \dfrac{x - y + z}{2x + y - z}$, $(1, 0, -2)$

In Exercises 65–68, find and sketch the domain of f. Then find an equation for the level curve or surface of the function passing through the given point.

65. $f(x, y) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n$, $(1, 2)$

66. $g(x, y, z) = \displaystyle\sum_{n=0}^{\infty} \frac{(x + y)^n}{n! z^n}$, $(\ln 4, \ln 9, 2)$

67. $f(x, y) = \displaystyle\int_x^y \frac{d\theta}{\sqrt{1 - \theta^2}}$, $(0, 1)$

68. $g(x, y, z) = \displaystyle\int_x^y \frac{dt}{1 + t^2} + \int_0^z \frac{d\theta}{\sqrt{4 - \theta^2}}$, $\left(0, 1, \sqrt{3}\right)$

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for each of the functions in Exercises 69–72.

a. Plot the surface over the given rectangle.

b. Plot several level curves in the rectangle.

c. Plot the level curve of f through the given point.

69. $f(x, y) = x \sin \dfrac{y}{2} + y \sin 2x$, $0 \le x \le 5\pi$, $0 \le y \le 5\pi$, $P(3\pi, 3\pi)$

70. $f(x, y) = (\sin x)(\cos y)e^{\sqrt{x^2+y^2}/8}$, $0 \le x \le 5\pi$, $0 \le y \le 5\pi$, $P(4\pi, 4\pi)$

71. $f(x, y) = \sin(x + 2\cos y)$, $-2\pi \le x \le 2\pi$, $-2\pi \le y \le 2\pi$, $P(\pi, \pi)$

72. $f(x, y) = e^{(x^{0.1} - y)} \sin(x^2 + y^2)$, $0 \le x \le 2\pi$, $-2\pi \le y \le \pi$, $P(\pi, -\pi)$

Use a CAS to plot the implicitly defined level surfaces in Exercises 73–76.

73. $4 \ln(x^2 + y^2 + z^2) = 1$

74. $x^2 + z^2 = 1$

75. $x + y^2 - 3z^2 = 1$

76. $\sin\left(\dfrac{x}{2}\right) - (\cos y)\sqrt{x^2 + z^2} = 2$

Parametrized Surfaces Just as you describe curves in the plane parametrically with a pair of equations $x = f(t)$, $y = g(t)$ defined on some parameter interval I, you can sometimes describe surfaces in space with a triple of equations $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$ defined on some parameter rectangle $a \le u \le b$, $c \le v \le d$. Many computer algebra systems permit you to plot such surfaces in *parametric mode*. (Parametrized surfaces are discussed in detail in Section 16.5.) Use a CAS to plot the surfaces in Exercises 77–80. Also plot several level curves in the xy-plane.

77. $x = u \cos v$, $y = u \sin v$, $z = u$, $0 \le u \le 2$, $0 \le v \le 2\pi$

78. $x = u \cos v$, $y = u \sin v$, $z = v$, $0 \le u \le 2$, $0 \le v \le 2\pi$

79. $x = (2 + \cos u) \cos v$, $y = (2 + \cos u) \sin v$, $z = \sin u$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$

80. $x = 2 \cos u \cos v$, $y = 2 \cos u \sin v$, $z = 2 \sin u$, $0 \le u \le 2\pi$, $0 \le v \le \pi$

14.2 Limits and Continuity in Higher Dimensions

This section treats limits and continuity for multivariable functions. These ideas are analogous to limits and continuity for single-variable functions, but including more independent variables leads to additional complexity and important differences requiring some new ideas.

Limits for Functions of Two Variables

If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0). This is similar to the informal definition for the limit of a function of a single variable. Notice, however, that if (x_0, y_0) lies in the interior of f's domain, (x, y) can approach (x_0, y_0) from any direction. For the limit to exist, the same limiting value must be obtained whatever direction of approach is taken. We illustrate this issue in several examples following the definition.

Whenever it is correctly defined, the composite of continuous functions is also continuous. The only requirement is that each function be continuous where it is applied. The proof, omitted here, is similar to that for functions of a single variable (Theorem 9 in Section 2.5).

> **Continuity of Composites**
>
> If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

For example, the composite functions

$$e^{x-y}, \qquad \cos \frac{xy}{x^2 + 1}, \qquad \ln\left(1 + x^2y^2\right)$$

are continuous at every point (x, y).

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln(x + y + z) \qquad \text{and} \qquad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P \to (1,0,-1)} \frac{e^{x+z}}{z^2 + \cos\sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z), may be found by direct substitution.

Extreme Values of Continuous Functions on Closed, Bounded Sets

The Extreme Value Theorem (Theorem 1, Section 4.1) states that a function of a single variable that is continuous throughout a closed, bounded interval $[a, b]$ takes on an absolute maximum value and an absolute minimum value at least once in $[a, b]$. The same holds true of a function $z = f(x, y)$ that is continuous on a closed, bounded set R in the plane (like a line segment, a disk, or a filled-in triangle). The function takes on an absolute maximum value at some point in R and an absolute minimum value at some point in R. The function may take on a maximum or minimum value more than once over R.

Similar results hold for functions of three or more variables. A continuous function $w = f(x, y, z)$, for example, must take on absolute maximum and minimum values on any closed, bounded set (solid ball or cube, spherical shell, rectangular solid) on which it is defined. We will learn how to find these extreme values in Section 14.7.

Exercises 14.2

Limits with Two Variables

Find the limits in Exercises 1–12.

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

2. $\displaystyle\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}}$

3. $\displaystyle\lim_{(x,y)\to(3,4)} \sqrt{x^2 + y^2 - 1}$

4. $\displaystyle\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2$

5. $\displaystyle\lim_{(x,y)\to(0,\pi/4)} \sec x \tan y$

6. $\displaystyle\lim_{(x,y)\to(0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$

7. $\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$

8. $\lim_{(x,y)\to(1,1)} \ln\left|1 + x^2y^2\right|$

9. $\lim_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x}$

10. $\lim_{(x,y)\to(1/27,\pi^3)} \cos\sqrt[3]{xy}$

11. $\lim_{(x,y)\to(1,\pi/6)} \dfrac{x \sin y}{x^2+1}$

12. $\lim_{(x,y)\to(\pi/2,0)} \dfrac{\cos y + 1}{y - \sin x}$

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

13. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - 2xy + y^2}{x - y}$

14. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - y^2}{x - y}$

15. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{xy - y - 2x + 2}{x - 1}$

16. $\lim_{\substack{(x,y)\to(2,-4)\\ x\neq -4,\, x\neq x^2}} \dfrac{y + 4}{x^2y - xy + 4x^2 - 4x}$

17. $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \dfrac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$

18. $\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \dfrac{x + y - 4}{\sqrt{x+y} - 2}$

19. $\lim_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \dfrac{\sqrt{2x-y} - 2}{2x - y - 4}$

20. $\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$

21. $\lim_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2 + y^2\right)}{x^2 + y^2}$

22. $\lim_{(x,y)\to(0,0)} \dfrac{1 - \cos(xy)}{xy}$

23. $\lim_{(x,y)\to(1,-1)} \dfrac{x^3 + y^3}{x + y}$

24. $\lim_{(x,y)\to(2,2)} \dfrac{x - y}{x^4 - y^4}$

Limits with Three Variables

Find the limits in Exercises 25–30.

25. $\lim_{P\to(1,3,4)} \left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$

26. $\lim_{P\to(1,-1,-1)} \dfrac{2xy + yz}{x^2 + z^2}$

27. $\lim_{P\to(\pi,\pi,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$

28. $\lim_{P\to(-1/4,\pi/2,2)} \tan^{-1} xyz$

29. $\lim_{P\to(\pi,0,3)} ze^{-2y}\cos 2x$

30. $\lim_{P\to(2,-3,6)} \ln\sqrt{x^2 + y^2 + z^2}$

Continuity for Two Variables

At what points (x, y) in the plane are the functions in Exercises 31–34 continuous?

31. a. $f(x, y) = \sin(x + y)$ b. $f(x, y) = \ln(x^2 + y^2)$

32. a. $f(x, y) = \dfrac{x + y}{x - y}$ b. $f(x, y) = \dfrac{y}{x^2 + 1}$

33. a. $g(x, y) = \sin\dfrac{1}{xy}$ b. $g(x, y) = \dfrac{x + y}{2 + \cos x}$

34. a. $g(x, y) = \dfrac{x^2 + y^2}{x^2 - 3x + 2}$ b. $g(x, y) = \dfrac{1}{x^2 - y}$

Continuity for Three Variables

At what points (x, y, z) in space are the functions in Exercises 35–40 continuous?

35. a. $f(x, y, z) = x^2 + y^2 - 2z^2$
 b. $f(x, y, z) = \sqrt{x^2 + y^2 - 1}$

36. a. $f(x, y, z) = \ln xyz$ b. $f(x, y, z) = e^{x+y}\cos z$

37. a. $h(x, y, z) = xy \sin\dfrac{1}{z}$ b. $h(x, y, z) = \dfrac{1}{x^2 + z^2 - 1}$

38. a. $h(x, y, z) = \dfrac{1}{|y| + |z|}$ b. $h(x, y, z) = \dfrac{1}{|xy| + |z|}$

39. a. $h(x, y, z) = \ln(z - x^2 - y^2 - 1)$
 b. $h(x, y, z) = \dfrac{1}{z - \sqrt{x^2 + y^2}}$

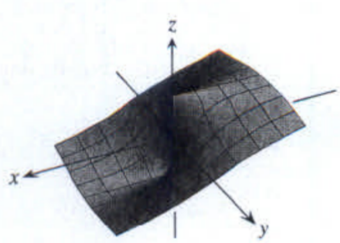
40. a. $h(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$
 b. $h(x, y, z) = \dfrac{1}{4 - \sqrt{x^2 + y^2 + z^2 - 9}}$

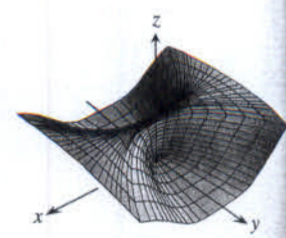
No Limit Exists at the Origin

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \to (0, 0)$.

41. $f(x, y) = -\dfrac{x}{\sqrt{x^2 + y^2}}$

42. $f(x, y) = \dfrac{x^4}{x^4 + y^2}$





43. $f(x, y) = \dfrac{x^4 - y^2}{x^4 + y^2}$

44. $f(x, y) = \dfrac{xy}{|xy|}$

45. $g(x, y) = \dfrac{x - y}{x + y}$

46. $g(x, y) = \dfrac{x^2 - y}{x - y}$

47. $h(x, y) = \dfrac{x^2 + y}{y}$

48. $h(x, y) = \dfrac{x^2y}{x^4 + y^2}$

Theory and Examples

In Exercises 49 and 50, show that the limits do not exist.

49. $\lim_{(x,y)\to(1,1)} \dfrac{xy^2 - 1}{y - 1}$

50. $\lim_{(x,y)\to(1,-1)} \dfrac{xy + 1}{x^2 - y^2}$

51. Let $f(x, y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Find each of the following limits, or explain that the limit does not exist.

a. $\lim_{(x,y)\to(0,1)} f(x, y)$

b. $\lim_{(x,y)\to(2,3)} f(x, y)$

c. $\lim_{(x,y)\to(0,0)} f(x, y)$

52. Let $f(x, y) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}$.

Find the following limits.

a. $\lim_{(x, y) \to (3, -2)} f(x, y)$

b. $\lim_{(x, y) \to (-2, 1)} f(x, y)$

c. $\lim_{(x, y) \to (0,0)} f(x, y)$

53. Show that the function in Example 6 has limit 0 along every straight line approaching (0, 0).

54. If $f(x_0, y_0) = 3$, what can you say about

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y)$$

if f is continuous at (x_0, y_0)? If f is not continuous at (x_0, y_0)? Give reasons for your answers.

The **Sandwich Theorem** for functions of two variables states that if $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L.$$

Use this result to support your answers to the questions in Exercises 55–58.

55. Does knowing that

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

tell you anything about

$$\lim_{(x, y) \to (0,0)} \frac{\tan^{-1} xy}{xy}?$$

Give reasons for your answer.

56. Does knowing that

$$2|xy| - \frac{x^2y^2}{6} < 4 - 4 \cos \sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x, y) \to (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|}?$$

Give reasons for your answer.

57. Does knowing that $|\sin (1/x)| \leq 1$ tell you anything about

$$\lim_{(x, y) \to (0,0)} y \sin \frac{1}{x}?$$

Give reasons for your answer.

58. Does knowing that $|\cos (1/y)| \leq 1$ tell you anything about

$$\lim_{(x, y) \to (0,0)} x \cos \frac{1}{y}?$$

Give reasons for your answer.

59. (*Continuation of Example 5.*)

a. Reread Example 5. Then substitute $m = \tan \theta$ into the formula

$$f(x, y)\bigg|_{y=mx} = \frac{2m}{1 + m^2}$$

and simplify the result to show how the value of f varies with the line's angle of inclination.

b. Use the formula you obtained in part (a) to show that the limit of f as $(x, y) \to (0, 0)$ along the line $y = mx$ varies from -1 to 1 depending on the angle of approach.

60. **Continuous extension** Define $f(0, 0)$ in a way that extends

$$f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

to be continuous at the origin.

Changing Variables to Polar Coordinates

If you cannot make any headway with $\lim_{(x, y) \to (0,0)} f(x, y)$ in rectangular coordinates, try changing to polar coordinates. Substitute $x = r \cos \theta$, $y = r \sin \theta$, and investigate the limit of the resulting expression as $r \to 0$. In other words, try to decide whether there exists a number L satisfying the following criterion:

Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad \Rightarrow \quad |f(r, \theta) - L| < \epsilon. \tag{1}$$

If such an L exists, then

$$\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{r \to 0} f(r \cos \theta, r \sin \theta) = L.$$

For instance,

$$\lim_{(x, y) \to (0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r \to 0} \frac{r^3 \cos^3 \theta}{r^2} = \lim_{r \to 0} r \cos^3 \theta = 0.$$

To verify the last of these equalities, we need to show that Equation (1) is satisfied with $f(r, \theta) = r \cos^3 \theta$ and $L = 0$. That is, we need to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad \Rightarrow \quad |r \cos^3 \theta - 0| < \epsilon.$$

Since

$$|r \cos^3 \theta| = |r||\cos^3 \theta| \leq |r| \cdot 1 = |r|,$$

the implication holds for all r and θ if we take $\delta = \epsilon$.

In contrast,

$$\frac{x^2}{x^2 + y^2} = \frac{r^2 \cos^2 \theta}{r^2} = \cos^2 \theta$$

takes on all values from 0 to 1 regardless of how small $|r|$ is, so that $\lim_{(x, y) \to (0,0)} x^2/(x^2 + y^2)$ does not exist.

In each of these instances, the existence or nonexistence of the limit as $r \to 0$ is fairly clear. Shifting to polar coordinates does not always help, however, and may even tempt us to false conclusions. For example, the limit may exist along every straight line (or ray) $\theta =$ constant and yet fail to exist in the broader sense. Example 5 illustrates this point. In polar coordinates, $f(x, y) = (2x^2y)/(x^4 + y^2)$ becomes

$$f(r \cos \theta, r \sin \theta) = \frac{r \cos \theta \sin 2\theta}{r^2 \cos^4 \theta + \sin^2 \theta}$$

for $r \neq 0$. If we hold θ constant and let $r \to 0$, the limit is 0. On the path $y = x^2$, however, we have $r \sin \theta = r^2 \cos^2 \theta$ and

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + (r\cos^2\theta)^2}$$

$$= \frac{2r\cos^2\theta\sin\theta}{2r^2\cos^4\theta} = \frac{r\sin\theta}{r^2\cos^2\theta} = 1.$$

In Exercises 61–66, find the limit of f as $(x, y) \to (0, 0)$ or show that the limit does not exist.

61. $f(x, y) = \dfrac{x^3 - xy^2}{x^2 + y^2}$

62. $f(x, y) = \cos\left(\dfrac{x^3 - y^3}{x^2 + y^2}\right)$

63. $f(x, y) = \dfrac{y^2}{x^2 + y^2}$

64. $f(x, y) = \dfrac{2x}{x^2 + x + y^2}$

65. $f(x, y) = \tan^{-1}\left(\dfrac{|x| + |y|}{x^2 + y^2}\right)$

66. $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

In Exercises 67 and 68, define $f(0, 0)$ in a way that extends f to be continuous at the origin.

67. $f(x, y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right)$

68. $f(x, y) = \dfrac{3x^2y}{x^2 + y^2}$

Using the Limit Definition

Each of Exercises 69–74 gives a function $f(x, y)$ and a positive number ϵ. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y),

$$\sqrt{x^2 + y^2} < \delta \quad \Rightarrow \quad |f(x, y) - f(0, 0)| < \epsilon.$$

69. $f(x, y) = x^2 + y^2, \quad \epsilon = 0.01$

70. $f(x, y) = y/(x^2 + 1), \quad \epsilon = 0.05$

71. $f(x, y) = (x + y)/(x^2 + 1), \quad \epsilon = 0.01$

72. $f(x, y) = (x + y)/(2 + \cos x), \quad \epsilon = 0.02$

73. $f(x, y) = \dfrac{xy^2}{x^2 + y^2}$ and $f(0, 0) = 0, \quad \epsilon = 0.04$

74. $f(x, y) = \dfrac{x^3 + y^4}{x^2 + y^2}$ and $f(0, 0) = 0, \quad \epsilon = 0.02$

Each of Exercises 75–78 gives a function $f(x, y, z)$ and a positive number ϵ. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y, z),

$$\sqrt{x^2 + y^2 + z^2} < \delta \quad \Rightarrow \quad |f(x, y, z) - f(0, 0, 0)| < \epsilon.$$

75. $f(x, y, z) = x^2 + y^2 + z^2, \quad \epsilon = 0.015$

76. $f(x, y, z) = xyz, \quad \epsilon = 0.008$

77. $f(x, y, z) = \dfrac{x + y + z}{x^2 + y^2 + z^2 + 1}, \quad \epsilon = 0.015$

78. $f(x, y, z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \epsilon = 0.03$

79. Show that $f(x, y, z) = x + y - z$ is continuous at every point (x_0, y_0, z_0).

80. Show that $f(x, y, z) = x^2 + y^2 + z^2$ is continuous at the origin.

14.3 Partial Derivatives

The calculus of several variables is similar to single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a "partial" derivative. This section shows how partial derivatives are defined and interpreted geometrically, and how to calculate them by applying the rules for differentiating functions of a single variable. The idea of *differentiability* for functions of several variables requires more than the existence of the partial derivatives because a point can be approached from so many different directions. However, we will see that differentiable functions of several variables behave in the same way as differentiable single-variable functions, so they are continuous and can be well approximated by linear functions.

Partial Derivatives of a Function of Two Variables

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$ (Figure 14.16). This curve is the graph of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x; the vertical coordinate is z. The y-value is held constant at y_0, so y is not a variable.

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$. To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than the d previously used. In the definition, h represents a real number, positive or negative.

As we can see from Corollary 3 and Theorem 4, a function $f(x, y)$ must be continuous at a point (x_0, y_0) if f_x and f_y are continuous throughout an open region containing (x_0, y_0). Remember, however, that it is still possible for a function of two variables to be discontinuous at a point where its first partial derivatives exist, as we saw in Example 8. Existence alone of the partial derivatives at that point is not enough, but continuity of the partial derivatives guarantees differentiability.

Exercises 14.3

Calculating First-Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

1. $f(x, y) = 2x^2 - 3y - 4$
2. $f(x, y) = x^2 - xy + y^2$
3. $f(x, y) = (x^2 - 1)(y + 2)$
4. $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
5. $f(x, y) = (xy - 1)^2$
6. $f(x, y) = (2x - 3y)^3$
7. $f(x, y) = \sqrt{x^2 + y^2}$
8. $f(x, y) = (x^3 + (y/2))^{2/3}$
9. $f(x, y) = 1/(x + y)$
10. $f(x, y) = x/(x^2 + y^2)$
11. $f(x, y) = (x + y)/(xy - 1)$
12. $f(x, y) = \tan^{-1}(y/x)$
13. $f(x, y) = e^{(x+y+1)}$
14. $f(x, y) = e^{-x}\sin(x + y)$
15. $f(x, y) = \ln(x + y)$
16. $f(x, y) = e^{xy}\ln y$
17. $f(x, y) = \sin^2(x - 3y)$
18. $f(x, y) = \cos^2(3x - y^2)$
19. $f(x, y) = x^y$
20. $f(x, y) = \log_y x$
21. $f(x, y) = \int_x^y g(t)\,dt$ (g continuous for all t)
22. $f(x, y) = \sum_{n=0}^{\infty}(xy)^n$ $(|xy| < 1)$

In Exercises 23–34, find f_x, f_y, and f_z.

23. $f(x, y, z) = 1 + xy^2 - 2z^2$
24. $f(x, y, z) = xy + yz + xz$
25. $f(x, y, z) = x - \sqrt{y^2 + z^2}$
26. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
27. $f(x, y, z) = \sin^{-1}(xyz)$
28. $f(x, y, z) = \sec^{-1}(x + yz)$
29. $f(x, y, z) = \ln(x + 2y + 3z)$
30. $f(x, y, z) = yz\ln(xy)$
31. $f(x, y, z) = e^{-(x^2+y^2+z^2)}$
32. $f(x, y, z) = e^{-xyz}$
33. $f(x, y, z) = \tanh(x + 2y + 3z)$
34. $f(x, y, z) = \sinh(xy - z^2)$

In Exercises 35–40, find the partial derivative of the function with respect to each variable.

35. $f(t, \alpha) = \cos(2\pi t - \alpha)$
36. $g(u, v) = v^2 e^{(2u/v)}$
37. $h(\rho, \phi, \theta) = \rho\sin\phi\cos\theta$
38. $g(r, \theta, z) = r(1 - \cos\theta) - z$
39. **Work done by the heart** (Section 3.11, Exercise 61)
$$W(P, V, \delta, v, g) = PV + \frac{V\delta v^2}{2g}$$
40. **Wilson lot size formula** (Section 4.6, Exercise 53)
$$A(c, h, k, m, q) = \frac{km}{q} + cm + \frac{hq}{2}$$

Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–50.

41. $f(x, y) = x + y + xy$
42. $f(x, y) = \sin xy$
43. $g(x, y) = x^2y + \cos y + y\sin x$
44. $h(x, y) = xe^y + y + 1$
45. $r(x, y) = \ln(x + y)$
46. $s(x, y) = \tan^{-1}(y/x)$
47. $w = x^2\tan(xy)$
48. $w = ye^{x^2-y}$
49. $w = x\sin(x^2y)$
50. $w = \dfrac{x - y}{x^2 + y}$

Mixed Partial Derivatives

In Exercises 51–54, verify that $w_{xy} = w_{yx}$.

51. $w = \ln(2x + 3y)$
52. $w = e^x + x\ln y + y\ln x$
53. $w = xy^2 + x^2y^3 + x^3y^4$
54. $w = x\sin y + y\sin x + xy$
55. Which order of differentiation will calculate f_{xy} faster: x first or y first? Try to answer without writing anything down.
    a. $f(x, y) = x\sin y + e^y$
    b. $f(x, y) = 1/x$
    c. $f(x, y) = y + (x/y)$
    d. $f(x, y) = y + x^2y + 4y^3 - \ln(y^2 + 1)$
    e. $f(x, y) = x^2 + 5xy + \sin x + 7e^x$
    f. $f(x, y) = x\ln xy$
56. The fifth-order partial derivative $\partial^5 f/\partial x^2\partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y? Try to answer without writing anything down.
    a. $f(x, y) = y^2x^4e^x + 2$
    b. $f(x, y) = y^2 + y(\sin x - x^4)$
    c. $f(x, y) = x^2 + 5xy + \sin x + 7e^x$
    d. $f(x, y) = xe^{y^2/2}$

Using the Partial Derivative Definition

In Exercises 57–60, use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

57. $f(x, y) = 1 - x + y - 3x^2y, \quad \frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(1, 2)$

58. $f(x, y) = 4 + 2x - 3y - xy^2, \quad \frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(-2, 1)$

59. $f(x, y) = \sqrt{2x + 3y - 1}, \quad \frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(-2, 3)$

60. $f(x, y) = \begin{cases} \dfrac{\sin(x^3 + y^4)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0), \end{cases}$

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(0, 0)$

61. Let $f(x, y) = 2x + 3y - 4$. Find the slope of the line tangent to this surface at the point $(2, -1)$ and lying in the **a.** plane $x = 2$ **b.** plane $y = -1$.

62. Let $f(x, y) = x^2 + y^3$. Find the slope of the line tangent to this surface at the point $(-1, 1)$ and lying in the **a.** plane $x = -1$ **b.** plane $y = 1$.

63. Three variables Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f/\partial z$ at (x_0, y_0, z_0). Use this definition to find $\partial f/\partial z$ at $(1, 2, 3)$ for $f(x, y, z) = x^2yz^2$.

64. Three variables Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f/\partial y$ at (x_0, y_0, z_0). Use this definition to find $\partial f/\partial y$ at $(-1, 0, 3)$ for $f(x, y, z) = -2xy^2 + yz^2$.

Differentiating Implicitly

65. Find the value of $\partial z/\partial x$ at the point $(1, 1, 1)$ if the equation

$$xy + z^3x - 2yz = 0$$

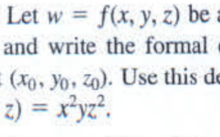
defines z as a function of the two independent variables x and y and the partial derivative exists.

66. Find the value of $\partial x/\partial z$ at the point $(1, -1, -3)$ if the equation

$$xz + y \ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

Exercises 67 and 68 are about the triangle shown here.



67. Express A implicitly as a function of a, b, and c and calculate $\partial A/\partial a$ and $\partial A/\partial b$.

68. Express a implicitly as a function of A, b, and B and calculate $\partial a/\partial A$ and $\partial a/\partial B$.

69. Two dependent variables Express v_x in terms of u and y if the equations $x = v \ln u$ and $y = u \ln v$ define u and v as functions of the independent variables x and y, and if v_x exists. (*Hint:* Differentiate both equations with respect to x and solve for v_x by eliminating u_x.)

70. Two dependent variables Find $\partial x/\partial u$ and $\partial y/\partial u$ if the equations $u = x^2 - y^2$ and $v = x^2 - y$ define x and y as functions of the independent variables u and v, and the partial derivatives exist. (See the hint in Exercise 69.) Then let $s = x^2 + y^2$ and find $\partial s/\partial u$.

Theory and Examples

71. Let $f(x, y) = \begin{cases} y^3, & y \geq 0 \\ -y^2, & y < 0. \end{cases}$

Find f_x, f_y, f_{xy}, and f_{yx}, and state the domain for each partial derivative.

72. Let $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq 0, \\ 0, & \text{if } (x, y) = 0. \end{cases}$

a. Show that $\frac{\partial f}{\partial y}(x, 0) = x$ for all x, and $\frac{\partial f}{\partial x}(0, y) = -y$ for all y.

b. Show that $\frac{\partial^2 f}{\partial y \partial x}(0, 0) \neq \frac{\partial^2 f}{\partial x \partial y}(0, 0)$.

The graph of f is shown on page 800.

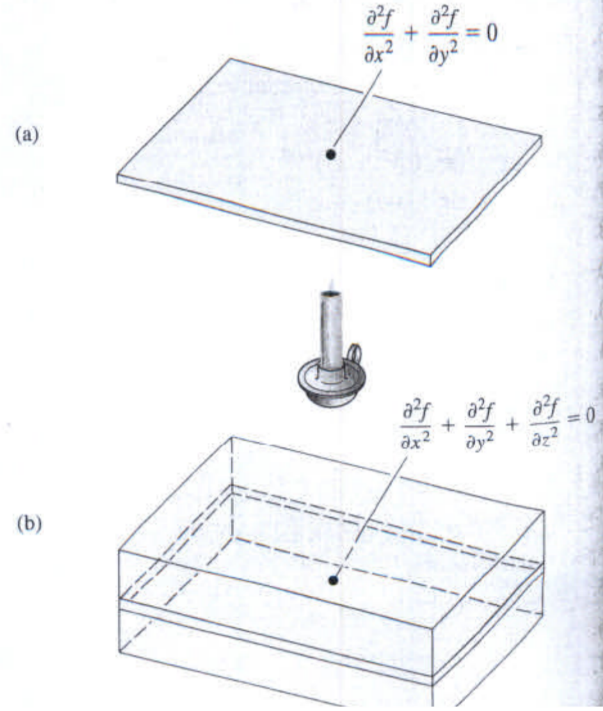
The **three-dimensional Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is satisfied by steady-state temperature distributions $T = f(x, y, z)$ in space, by gravitational potentials, and by electrostatic potentials. The **two-dimensional Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

obtained by dropping the $\partial^2 f/\partial z^2$ term from the previous equation, describes potentials and steady-state temperature distributions in a plane (see the accompanying figure). The plane (a) may be treated as a thin slice of the solid (b) perpendicular to the z-axis.



Show that each function in Exercises 73–80 satisfies a Laplace equation.

73. $f(x, y, z) = x^2 + y^2 - 2z^2$

74. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$

75. $f(x, y) = e^{-2y} \cos 2x$

76. $f(x, y) = \ln \sqrt{x^2 + y^2}$

77. $f(x, y) = 3x + 2y - 4$

78. $f(x, y) = \tan^{-1} \frac{x}{y}$

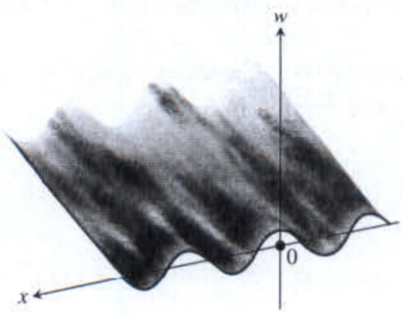
79. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

80. $f(x, y, z) = e^{3x+4y} \cos 5z$

The Wave Equation If we stand on an ocean shore and take a snapshot of the waves, the picture shows a regular pattern of peaks and valleys in an instant of time. We see periodic vertical motion in space, with respect to distance. If we stand in the water, we can feel the rise and fall of the water as the waves go by. We see periodic vertical motion in time. In physics, this beautiful symmetry is expressed by the **one-dimensional wave equation**

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},$$

where w is the wave height, x is the distance variable, t is the time variable, and c is the velocity with which the waves are propagated.



In our example, x is the distance across the ocean's surface, but in other applications, x might be the distance along a vibrating string, distance through air (sound waves), or distance through space (light waves). The number c varies with the medium and type of wave.

Show that the functions in Exercises 81–87 are all solutions of the wave equation.

81. $w = \sin (x + ct)$

82. $w = \cos (2x + 2ct)$

83. $w = \sin (x + ct) + \cos (2x + 2ct)$

84. $w = \ln (2x + 2ct)$

85. $w = \tan (2x - 2ct)$

86. $w = 5 \cos (3x + 3ct) + e^{x+ct}$

87. $w = f(u)$, where f is a differentiable function of u, and $u = a(x + ct)$, where a is a constant

88. Does a function $f(x, y)$ with continuous first partial derivatives throughout an open region R have to be continuous on R? Give reasons for your answer.

89. If a function $f(x, y)$ has continuous second partial derivatives throughout an open region R, must the first-order partial derivatives of f be continuous on R? Give reasons for your answer.

90. The heat equation An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the *one-dimensional heat equation*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

Show that $u(x, t) = \sin (\alpha x) \cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?

91. Let $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$. (*Hint:* Use Theorem 4 and show that f is not continuous at $(0, 0)$.)

92. Let $f(x, y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \text{otherwise.} \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

14.4 The Chain Rule

The Chain Rule for functions of a single variable studied in Section 3.6 says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, w is a differentiable function of t and dw/dt can be calculated by the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

For this composite function $w(t) = f(g(t))$, we can think of t as the independent variable and $x = g(t)$ as the "intermediate variable," because t determines the value of x which in turn gives the value of w from the function f. We display the Chain Rule in a "branch diagram" in the margin on the next page.

For functions of several variables the Chain Rule has more than one form, which depends on how many independent and intermediate variables are involved. However, once the variables are taken into account, the Chain Rule works in the same way we just discussed.

Since $F(0, 0, 0) = 0$, $F_z(0, 0, 0) = 1 \neq 0$, and all first partial derivatives are continuous, the Implicit Function Theorem says that $F(x, y, z) = 0$ defines z as a differentiable function of x and y near the point $(0, 0, 0)$. From Equations (2),

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z \sin y}{2z + xye^{xz} + \cos y}.$$

At $(0, 0, 0)$ we find

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$

■

Functions of Many Variables

We have seen several different forms of the Chain Rule in this section, but each one is just a special case of one general formula. When solving particular problems, it may help to draw the appropriate branch diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom. To find the derivative of the dependent variable with respect to the selected independent variable, start at the dependent variable and read down each route of the branch diagram to the independent variable, calculating and multiplying the derivatives along each route. Then add the products found for the different routes.

In general, suppose that $w = f(x, y, \dots, v)$ is a differentiable function of the intermediate variables $x, y, \dots, v$ (a finite set) and the $x, y, \dots, v$ are differentiable functions of the independent variables $p, q, \dots, t$ (another finite set). Then w is a differentiable function of the variables p through t, and the partial derivatives of w with respect to these variables are given by equations of the form

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}.$$

The other equations are obtained by replacing p by $q, \dots, t$, one at a time.

One way to remember this equation is to think of the right-hand side as the dot product of two vectors with components

$$\underbrace{\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \dots, \frac{\partial w}{\partial v}\right)}_{\text{Derivatives of } w \text{ with respect to the intermediate variables}} \quad \text{and} \quad \underbrace{\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \dots, \frac{\partial v}{\partial p}\right)}_{\text{Derivatives of the intermediate variables with respect to the selected independent variable}}.$$

Exercises 14.4

Chain Rule: One Independent Variable

In Exercises 1–6, **(a)** express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then **(b)** evaluate dw/dt at the given value of t.

1. $w = x^2 + y^2$, $x = \cos t$, $y = \sin t$; $t = \pi$

2. $w = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$; $t = 0$

3. $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = 1/t$; $t = 3$

4. $w = \ln(x^2 + y^2 + z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$; $t = 3$

5. $w = 2ye^x - \ln z$, $x = \ln(t^2 + 1)$, $y = \tan^{-1} t$, $z = e^t$; $t = 1$

6. $w = z - \sin xy$, $x = t$, $y = \ln t$, $z = e^{t-1}$; $t = 1$

Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

7. $z = 4e^x \ln y$, $x = \ln(u \cos v)$, $y = u \sin v$; $(u, v) = (2, \pi/4)$

8. $z = \tan^{-1}(x/y), \quad x = u\cos v, \quad y = u \sin v;$
$(u, v) = (1.3, \pi/6)$

In Exercises 9 and 10, **(a)** express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of u and v both by using the Chain Rule and by expressing w directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point (u, v).

9. $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv;$
$(u, v) = (1/2, 1)$

10. $w = \ln(x^2 + y^2 + z^2), \quad x = ue^v \sin u, \quad y = ue^v \cos u,$
$z = ue^v; \quad (u, v) = (-2, 0)$

In Exercises 11 and 12, **(a)** express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of x, y, and z both by using the Chain Rule and by expressing u directly in terms of x, y, and z before differentiating. Then **(b)** evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point (x, y, z).

11. $u = \dfrac{p - q}{q - r}, \quad p = x + y + z, \quad q = x - y + z,$
$r = x + y - z; \quad (x, y, z) = \left(\sqrt{3}, 2, 1\right)$

12. $u = e^{qr}\sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = 1/z;$
$(x, y, z) = (\pi/4, 1/2, -1/2)$

Using a Branch Diagram

In Exercises 13–24, draw a branch diagram and write a Chain Rule formula for each derivative.

13. $\dfrac{dz}{dt}$ for $z = f(x, y), \quad x = g(t), \quad y = h(t)$

14. $\dfrac{dz}{dt}$ for $z = f(u, v, w), \quad u = g(t), \quad v = h(t), \quad w = k(t)$

15. $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ for $w = h(x, y, z), \quad x = f(u, v), \quad y = g(u, v),$
$z = k(u, v)$

16. $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ for $w = f(r, s, t), \quad r = g(x, y), \quad s = h(x, y),$
$t = k(x, y)$

17. $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ for $w = g(x, y), \quad x = h(u, v), \quad y = k(u, v)$

18. $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ for $w = g(u, v), \quad u = h(x, y), \quad v = k(x, y)$

19. $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial s}$ for $z = f(x, y), \quad x = g(t, s), \quad y = h(t, s)$

20. $\dfrac{\partial y}{\partial r}$ for $y = f(u), \quad u = g(r, s)$

21. $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$ for $w = g(u), \quad u = h(s, t)$

22. $\dfrac{\partial w}{\partial p}$ for $w = f(x, y, z, v), \quad x = g(p, q), \quad y = h(p, q),$
$z = j(p, q), \quad v = k(p, q)$

23. $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ for $w = f(x, y), \quad x = g(r), \quad y = h(s)$

24. $\dfrac{\partial w}{\partial s}$ for $w = g(x, y), \quad x = h(r, s, t), \quad y = k(r, s, t)$

Implicit Differentiation

Assuming that the equations in Exercises 25–28 define y as a differentiable function of x, use Theorem 8 to find the value of dy/dx at the given point.

25. $x^3 - 2y^2 + xy = 0, \quad (1, 1)$

26. $xy + y^2 - 3x - 3 = 0, \quad (-1, 1)$

27. $x^2 + xy + y^2 - 7 = 0, \quad (1, 2)$

28. [illegible]

Find the values of $\partial z/\partial x$ and $\partial z/\partial y$ at the points in Exercises 29–32.

29. $z^3 - xy + yz + y^3 - 2 = 0, \quad (1, 1, 1)$

30. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 1 = 0, \quad (2, 3, 6)$

31. $\sin(x + y) + \sin(y + z) + \sin(x + z) = 0, \quad (\pi, \pi, \pi)$

32. $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0, \quad (1, \ln 2, \ln 3)$

Finding Partial Derivatives at Specified Points

33. Find $\partial w/\partial r$ when $r = 1, s = -1$ if $w = (x + y + z)^2$, $x = r - s$, $y = \cos(r + s)$, $z = \sin(r + s)$.

34. Find $\partial w/\partial v$ when $u = -1, v = 2$ if $w = xy + \ln z$, $x = v^2/u$, $y = u + v$, $z = \cos u$.

35. Find $\partial w/\partial v$ when $u = 0, v = 0$ if $w = x^2 + (y/x)$, $x = u - 2v + 1$, $y = 2u + v - 2$.

36. Find $\partial z/\partial u$ when $u = 0, v = 1$ if $z = \sin xy + x\sin y$, $x = u^2 + v^2$, $y = uv$.

37. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = \ln 2$, $v = 1$ if $z = 5\tan^{-1} x$ and $x = e^u + \ln v$.

38. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = 1$, $v = -2$ if $z = \ln q$ and $q = \sqrt{v + 3}\tan^{-1} u$.

Theory and Examples

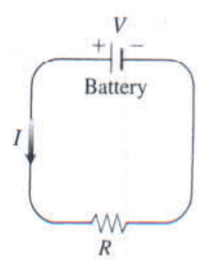
39. Assume that $w = f(s^3 + t^2)$ and $f'(x) = e^x$. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

40. Assume that $w = f\left(ts^2, \dfrac{s}{t}\right)$, $\dfrac{\partial f}{\partial x}(x, y) = xy$, and $\dfrac{\partial f}{\partial y}(x, y) = \dfrac{x^2}{2}$. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

41. **Changing voltage in a circuit** The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

to find how the current is changing at the instant when $R = 600$ ohms, $I = 0.04$ amp, $dR/dt = 0.5$ ohm/sec, and $dV/dt = -0.01$ volt/sec.



42. **Changing dimensions in a box** The lengths a, b, and c of the edges of a rectangular box are changing with time. At the instant in question, $a = 1$ m, $b = 2$ m, $c = 3$ m, $da/dt = db/dt = 1$ m/sec, and $dc/dt = -3$ m/sec. At what rates are the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

43. If $f(u, v, w)$ is differentiable and $u = x - y$, $v = y - z$, and $w = z - x$, show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$$

44. **Polar coordinates** Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $w = f(x, y)$.

a. Show that

$$\frac{\partial w}{\partial r} = f_x \cos\theta + f_y \sin\theta$$

and

$$\frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta.$$

b. Solve the equations in part (a) to express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial\theta$.

c. Show that

$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$$

45. **Laplace equations** Show that if $w = f(u, v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

46. **Laplace equations** Let $w = f(u) + g(v)$, where $u = x + iy$, $v = x - iy$, and $i = \sqrt{-1}$. Show that w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$ if all the necessary functions are differentiable.

47. **Extreme values on a helix** Suppose that the partial derivatives of a function $f(x, y, z)$ at points on the helix $x = \cos t$, $y = \sin t$, $z = t$ are

$$f_x = \cos t, \qquad f_y = \sin t, \qquad f_z = t^2 + t - 2.$$

At what points on the curve, if any, can f take on extreme values?

48. **A space curve** Let $w = x^2e^{2y}\cos 3z$. Find the value of dw/dt at the point $(1, \ln 2, 0)$ on the curve $x = \cos t$, $y = \ln(t + 2)$, $z = t$.

49. **Temperature on a circle** Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \qquad \frac{\partial T}{\partial y} = 8y - 4x.$$

a. Find where the maximum and minimum temperatures on the circle occur by examining the derivatives dT/dt and d^2T/dt^2.

b. Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of T on the circle.

50. **Temperature on an ellipse** Let $T = g(x, y)$ be the temperature at the point (x, y) on the ellipse

$$x = 2\sqrt{2}\cos t, \qquad y = \sqrt{2}\sin t, \qquad 0 \le t \le 2\pi,$$

and suppose that

$$\frac{\partial T}{\partial x} = y, \qquad \frac{\partial T}{\partial y} = x.$$

a. Locate the maximum and minimum temperatures on the ellipse by examining dT/dt and d^2T/dt^2.

b. Suppose that $T = xy - 2$. Find the maximum and minimum values of T on the ellipse.

Differentiating Integrals Under mild continuity restrictions, it is true that if

$$F(x) = \int_a^b g(t, x)\,dt,$$

then $F'(x) = \int_a^b g_x(t, x)\,dt$. Using this fact and the Chain Rule, we can find the derivative of

$$F(x) = \int_a^{f(x)} g(t, x)\,dt$$

by letting

$$G(u, x) = \int_a^u g(t, x)\,dt,$$

where $u = f(x)$. Find the derivatives of the functions in Exercises 51 and 52.

51. $F(x) = \int_0^{x^2} \sqrt{t^4 + x^3}\,dt$

52. $F(x) = \int_{x^2}^1 \sqrt{t^3 + x^2}\,dt$

14.5 Directional Derivatives and Gradient Vectors

If you look at the map (Figure 14.26) showing contours within Yosemite National Park in California, you will notice that the streams flow perpendicular to the contours. The streams are following paths of steepest descent so the waters reach lower elevations as quickly as possible. Therefore, the fastest instantaneous rate of change in a stream's elevation above sea level has a particular direction. In this section, you will see why this direction, called the "downhill" direction, is perpendicular to the contours.

Directional Derivatives in the Plane

We know from Section 14.4 that if $f(x, y)$ is differentiable, then the rate at which f changes with respect to t along a differentiable curve $x = g(t)$, $y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

At any point $P_0(x_0, y_0) = P_0(g(t_0), h(t_0))$, this equation gives the rate of change of f with respect to increasing t and therefore depends, among other things, on the direction of

Exercises 14.5

Calculating Gradients

In Exercises 1–6, find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $f(x, y) = y - x, \quad (2, 1)$

2. $f(x, y) = \ln(x^2 + y^2), \quad (1, 1)$

3. $g(x, y) = xy^2, \quad (2, -1)$

4. $g(x, y) = \frac{x^2}{2} - \frac{y^2}{2}, \quad (\sqrt{2}, 1)$

5. $f(x, y) = \sqrt{2x + 3y}, \quad (-1, 2)$

6. $f(x, y) = \tan^{-1}\frac{\sqrt{x}}{y}, \quad (4, -2)$

In Exercises 7–10, find ∇f at the given point.

7. $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x, \quad (1, 1, 1)$

8. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \tan^{-1} xz, \quad (1, 1, 1)$

9. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz), \quad (-1, 2, -2)$

10. $f(x, y, z) = e^{x+y}\cos z + (y + 1)\sin^{-1} x, \quad (0, 0, \pi/6)$

Finding Directional Derivatives

In Exercises 11–18, find the derivative of the function at P_0 in the direction of $\mathbf{u}$.

11. $f(x, y) = 2xy - 3y^2, \quad P_0(5, 5), \quad \mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$

12. $f(x, y) = 2x^2 + y^2, \quad P_0(-1, 1), \quad \mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$

13. $g(x, y) = \frac{x - y}{xy + 2}, \quad P_0(1, -1), \quad \mathbf{u} = 12\mathbf{i} + 5\mathbf{j}$

14. $h(x, y) = \tan^{-1}(y/x) + \sqrt{3}\sin^{-1}(xy/2), \quad P_0(1, 1),$
$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$

15. $f(x, y, z) = xy + yz + zx, \quad P_0(1, -1, 2), \quad \mathbf{u} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

16. $f(x, y, z) = x^2 + 2y^2 - 3z^2, \quad P_0(1, 1, 1), \quad \mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

17. $g(x, y, z) = 3e^x \cos yz, \quad P_0(0, 0, 0), \quad \mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

18. $h(x, y, z) = \cos xy + e^{yz} + \ln zx, \quad P_0(1, 0, 1/2),$
$\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

In Exercises 19–24, find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the functions in these directions.

19. $f(x, y) = x^2 + xy + y^2, \quad P_0(-1, 1)$

20. $f(x, y) = x^2 y + e^{xy}\sin y, \quad P_0(1, 0)$

21. $f(x, y, z) = (x/y) - yz, \quad P_0(4, 1, 1)$

22. $g(x, y, z) = xe^y + z^2, \quad P_0(1, \ln 2, 1/2)$

23. $f(x, y, z) = \ln xy + \ln yz + \ln xz, \quad P_0(1, 1, 1)$

24. $h(x, y, z) = \ln(x^2 + y^2 - 1) + y + 6z, \quad P_0(1, 1, 0)$

Tangent Lines to Level Curves

In Exercises 25–28, sketch the curve $f(x, y) = c$ together with ∇f and the tangent line at the given point. Then write an equation for the tangent line.

25. $x^2 + y^2 = 4, \quad (\sqrt{2}, \sqrt{2})$

26. $x^2 - y = 1, \quad (\sqrt{2}, 1)$

27. $xy = -4, \quad (2, -2)$

28. $x^2 - xy + y^2 = 7, \quad (-1, 2)$

Theory and Examples

29. Let $f(x, y) = x^2 - xy + y^2 - y$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f(1, -1)$ for which

a. $D_{\mathbf{u}} f(1, -1)$ is largest

b. $D_{\mathbf{u}} f(1, -1)$ is smallest

c. $D_{\mathbf{u}} f(1, -1) = 0$

d. $D_{\mathbf{u}} f(1, -1) = 4$

e. $D_{\mathbf{u}} f(1, -1) = -3$

30. Let $f(x, y) = \frac{(x - y)}{(x + y)}$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ for which

a. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is largest

b. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is smallest

c. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = 0$

d. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = -2$

e. $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = 1$

31. Zero directional derivative In what direction is the derivative of $f(x, y) = xy + y^2$ at $P(3, 2)$ equal to zero?

32. Zero directional derivative In what directions is the derivative of $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at $P(1, 1)$ equal to zero?

33. Is there a direction $\mathbf{u}$ in which the rate of change of $f(x, y) = x^2 - 3xy + 4y^2$ at $P(1, 2)$ equals 14? Give reasons for your answer.

34. Changing temperature along a circle Is there a direction $\mathbf{u}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ (temperature in degrees Celsius, distance in feet) at $P(1, -1, 1)$ is $-3°\text{C/ft}$? Give reasons for your answer.

35. The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of $\mathbf{i} + \mathbf{j}$ is $2\sqrt{2}$ and in the direction of $-2\mathbf{j}$ is -3. What is the derivative of f in the direction of $-\mathbf{i} - 2\mathbf{j}$? Give reasons for your answer.

36. The derivative of $f(x, y, z)$ at a point P is greatest in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. In this direction, the value of the derivative is $2\sqrt{3}$.

a. What is ∇f at P? Give reasons for your answer.

b. What is the derivative of f at P in the direction of $\mathbf{i} + \mathbf{j}$?

37. Directional derivatives and scalar components How is the derivative of a differentiable function $f(x, y, z)$ at a point P_0 in the direction of a unit vector $\mathbf{u}$ related to the scalar component of $(\nabla f)_{P_0}$ in the direction of $\mathbf{u}$? Give reasons for your answer.

38. Directional derivatives and partial derivatives Assuming that the necessary derivatives of $f(x, y, z)$ are defined, how are $D_{\mathbf{i}} f$, $D_{\mathbf{j}} f$, and $D_{\mathbf{k}} f$ related to f_x, f_y, and f_z? Give reasons for your answer.

39. Lines in the *xy*-plane Show that $A(x - x_0) + B(y - y_0) = 0$ is an equation for the line in the xy-plane through the point (x_0, y_0) normal to the vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$.

40. The algebra rules for gradients Given a constant k and the gradients

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \qquad \nabla g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k},$$

establish the algebra rules for gradients.

Functions of More Than Two Variables

Analogous results hold for differentiable functions of more than two variables.

1. The **linearization** of $f(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is
$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0).$$
2. Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of f are continuous. Suppose also that $|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R. Then the **error** $E(x, y, z) = f(x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by the inequality
$$|E| \le \frac{1}{2}M(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$
3. If the second partial derivatives of f are continuous and if x, y, and z change from x_0, y_0, and z_0 by small amounts dx, dy, and dz, the **total differential**
$$df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$$
gives a good approximation of the resulting change in f.

EXAMPLE 8 Find the linearization $L(x, y, z)$ of
$$f(x, y, z) = x^2 - xy + 3\sin z$$
at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangular region
$$R: \quad |x - 2| \le 0.01, \qquad |y - 1| \le 0.02, \qquad |z| \le 0.01.$$

Solution Routine calculations give
$$f(2, 1, 0) = 2, \qquad f_x(2, 1, 0) = 3, \qquad f_y(2, 1, 0) = -2, \qquad f_z(2, 1, 0) = 3.$$
Thus,
$$L(x, y, z) = 2 + 3(x - 2) + (-2)(y - 1) + 3(z - 0) = 3x - 2y + 3z - 2.$$
Since
$$f_{xx} = 2, \qquad f_{yy} = 0, \qquad f_{zz} = -3\sin z, \qquad f_{xy} = -1, \qquad f_{xz} = 0, \qquad f_{yz} = 0,$$
and $|-3\sin z| \le 3\sin 0.01 \approx 0.03$, we may take $M = 2$ as a bound on the second partials. Hence, the error incurred by replacing f by L on R satisfies
$$|E| \le \frac{1}{2}(2)(0.01 + 0.02 + 0.01)^2 = 0.0016.$$
■

Exercises 14.6

Tangent Planes and Normal Lines to Surfaces

In Exercises 1–8, find equations for the

(a) tangent plane and

(b) normal line at the point P_0 on the given surface.

1. $x^2 + y^2 + z^2 = 3, \quad P_0(1, 1, 1)$
2. $x^2 + y^2 - z^2 = 18, \quad P_0(3, 5, -4)$
3. $2z - x^2 = 0, \quad P_0(2, 0, 2)$
4. $x^2 + 2xy - y^2 + z^2 = 7, \quad P_0(1, -1, 3)$
5. $\cos \pi x - x^2y + e^{xz} + yz = 4, \quad P_0(0, 1, 2)$
6. $x^2 - xy - y^2 - z = 0, \quad P_0(1, 1, -1)$
7. $x + y + z = 1, \quad P_0(0, 1, 0)$
8. $x^2 + y^2 - 2xy - x + 3y - z = -4, \quad P_0(2, -3, 18)$

In Exercises 9–12, find an equation for the plane that is tangent to the given surface at the given point.

9. $z = \ln(x^2 + y^2), \quad (1, 0, 0)$
10. $z = e^{-(x^2+y^2)}, \quad (0, 0, 1)$
11. $z = \sqrt{y - x}, \quad (1, 2, 1)$
12. $z = 4x^2 + y^2, \quad (1, 1, 5)$

Tangent Lines to Intersecting Surfaces

In Exercises 13–18, find parametric equations for the line tangent to the curve of intersection of the surfaces at the given point.

13. Surfaces: $x + y^2 + 2z = 4, \quad x = 1$
Point: $(1, 1, 1)$

14. Surfaces: $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$
Point: $(1, 1, 1)$

15. Surfaces: $x^2 + 2y + 2z = 4, \quad y = 1$
Point: $(1, 1, 1/2)$

16. Surfaces: $x + y^2 + z = 2, \quad y = 1$
Point: $(1/2, 1, 1/2)$

17. Surfaces: $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0,$
$x^2 + y^2 + z^2 = 11$
Point: $(1, 1, 3)$

18. Surfaces: $x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0$
Point: $\left(\sqrt{2}, \sqrt{2}, 4\right)$

Estimating Change

19. By about how much will

$$f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$$

change if the point $P(x, y, z)$ moves from $P_0(3, 4, 12)$ a distance of $ds = 0.1$ unit in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$?

20. By about how much will

$$f(x, y, z) = e^x \cos yz$$

change as the point $P(x, y, z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction of $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$?

21. By about how much will

$$g(x, y, z) = x + x\cos z - y\sin z + y$$

change if the point $P(x, y, z)$ moves from $P_0(2, -1, 0)$ a distance of $ds = 0.2$ unit toward the point $P_1(0, 1, 2)$?

22. By about how much will

$$h(x, y, z) = \cos(\pi xy) + xz^2$$

change if the point $P(x, y, z)$ moves from $P_0(-1, -1, -1)$ a distance of $ds = 0.1$ unit toward the origin?

23. Temperature change along a circle Suppose that the Celsius temperature at the point (x, y) in the xy-plane is $T(x, y) = x \sin 2y$ and that distance in the xy-plane is measured in meters. A particle is moving *clockwise* around the circle of radius 1 m centered at the origin at the constant rate of 2 m/sec.

a. How fast is the temperature experienced by the particle changing in degrees Celsius per meter at the point $P\left(1/2, \sqrt{3}/2\right)$?

b. How fast is the temperature experienced by the particle changing in degrees Celsius per second at P?

24. Changing temperature along a space curve The Celsius temperature in a region in space is given by $T(x, y, z) = 2x^2 - xyz$. A particle is moving in this region and its position at time t is given by $x = 2t^2$, $y = 3t$, $z = -t^2$, where time is measured in seconds and distance in meters.

a. How fast is the temperature experienced by the particle changing in degrees Celsius per meter when the particle is at the point $P(8, 6, -4)$?

b. How fast is the temperature experienced by the particle changing in degrees Celsius per second at P?

Finding Linearizations

In Exercises 25–30, find the linearization $L(x, y)$ of the function at each point.

25. $f(x, y) = x^2 + y^2 + 1$ at **a.** $(0, 0)$, **b.** $(1, 1)$

26. $f(x, y) = (x + y + 2)^2$ at **a.** $(0, 0)$, **b.** $(1, 2)$

27. $f(x, y) = 3x - 4y + 5$ at **a.** $(0, 0)$, **b.** $(1, 1)$

28. $f(x, y) = x^3y^4$ at **a.** $(1, 1)$, **b.** $(0, 0)$

29. $f(x, y) = e^x \cos y$ at **a.** $(0, 0)$, **b.** $(0, \pi/2)$

30. $f(x, y) = e^{2y-x}$ at **a.** $(0, 0)$, **b.** $(1, 2)$

31. Wind chill factor Wind chill, a measure of the apparent temperature felt on exposed skin, is a function of air temperature and wind speed. The precise formula, updated by the National Weather Service in 2001 and based on modern heat transfer theory, a human face model, and skin tissue resistance, is

$$W = W(v, T) = 35.74 + 0.6215\,T - 35.75\,v^{0.16} + 0.4275\,T \cdot v^{0.16},$$

where T is air temperature in °F and v is wind speed in mph. A partial wind chill chart is given.

		T(°F)								
		30	**25**	**20**	**15**	**10**	**5**	**0**	**−5**	**−10**
v (mph)	**5**	25	19	13	7	1	−5	−11	−16	−22
	10	21	15	9	3	−4	−10	−16	−22	−28
	15	19	13	6	0	−7	−13	−19	−26	−32
	20	17	11	4	−2	−9	−15	−22	−29	−35
	25	16	9	3	−4	−11	−17	−24	−31	−37
	30	15	8	1	−5	−12	−19	−26	−33	−39
	35	14	7	0	−7	−14	−21	−27	−34	−41

a. Use the table to find $W(20, 25)$, $W(30, -10)$, and $W(15, 15)$.

b. Use the formula to find $W(10, -40)$, $W(50, -40)$, and $W(60, 30)$.

c. Find the linearization $L(v, T)$ of the function $W(v, T)$ at the point $(25, 5)$.

d. Use $L(v, T)$ in part (c) to estimate the following wind chill values.

i) $W(24, 6)$ **ii)** $W(27, 2)$

iii) $W(5, -10)$ (Explain why this value is much different from the value found in the table.)

32. Find the linearization $L(v, T)$ of the function $W(v, T)$ in Exercise 31 at the point $(50, -20)$. Use it to estimate the following wind chill values.

a. $W(49, -22)$

b. $W(53, -19)$

c. $W(60, -30)$

Bounding the Error in Linear Approximations

In Exercises 33–38, find the linearization $L(x, y)$ of the function $f(x, y)$ at P_0. Then find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x, y) \approx L(x, y)$ over the rectangle R.

33. $f(x, y) = x^2 - 3xy + 5$ at $P_0(2, 1)$,
$R: \ |x - 2| \le 0.1, \ |y - 1| \le 0.1$

34. $f(x, y) = (1/2)x^2 + xy + (1/4)y^2 + 3x - 3y + 4$ at $P_0(2, 2)$,
$R: \ |x - 2| \le 0.1, \ |y - 2| \le 0.1$

35. $f(x, y) = 1 + y + x \cos y$ at $P_0(0, 0)$,
$R: \ |x| \le 0.2, \ |y| \le 0.2$
(Use $|\cos y| \le 1$ and $|\sin y| \le 1$ in estimating E.)

36. $f(x, y) = xy^2 + y \cos(x - 1)$ at $P_0(1, 2)$,
$R: \ |x - 1| \le 0.1, \ |y - 2| \le 0.1$

37. $f(x, y) = e^x \cos y$ at $P_0(0, 0)$,
$R: \ |x| \le 0.1, \ |y| \le 0.1$
(Use $e^x \le 1.11$ and $|\cos y| \le 1$ in estimating E.)

38. $f(x, y) = \ln x + \ln y$ at $P_0(1, 1)$,
$R: \ |x - 1| \le 0.2, \ |y - 1| \le 0.2$

Linearizations for Three Variables

Find the linearizations $L(x, y, z)$ of the functions in Exercises 39–44 at the given points.

39. $f(x, y, z) = xy + yz + xz$ at
a. $(1, 1, 1)$ **b.** $(1, 0, 0)$ **c.** $(0, 0, 0)$

40. $f(x, y, z) = x^2 + y^2 + z^2$ at
a. $(1, 1, 1)$ **b.** $(0, 1, 0)$ **c.** $(1, 0, 0)$

41. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at
a. $(1, 0, 0)$ **b.** $(1, 1, 0)$ **c.** $(1, 2, 2)$

42. $f(x, y, z) = (\sin xy)/z$ at
a. $(\pi/2, 1, 1)$ **b.** $(2, 0, 1)$

43. $f(x, y, z) = e^x + \cos(y + z)$ at
a. $(0, 0, 0)$ **b.** $\left(0, \frac{\pi}{2}, 0\right)$ **c.** $\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)$

44. $f(x, y, z) = \tan^{-1}(xyz)$ at
a. $(1, 0, 0)$ **b.** $(1, 1, 0)$ **c.** $(1, 1, 1)$

In Exercises 45–48, find the linearization $L(x, y, z)$ of the function $f(x, y, z)$ at P_0. Then find an upper bound for the magnitude of the error E in the approximation $f(x, y, z) \approx L(x, y, z)$ over the region R.

45. $f(x, y, z) = xz - 3yz + 2$ at $P_0(1, 1, 2)$,
$R: \ |x - 1| \le 0.01, \ |y - 1| \le 0.01, \ |z - 2| \le 0.02$

46. $f(x, y, z) = x^2 + xy + yz + (1/4)z^2$ at $P_0(1, 1, 2)$,
$R: \ |x - 1| \le 0.01, \ |y - 1| \le 0.01, \ |z - 2| \le 0.08$

47. $f(x, y, z) = xy + 2yz - 3xz$ at $P_0(1, 1, 0)$,
$R: \ |x - 1| \le 0.01, \ |y - 1| \le 0.01, \ |z| \le 0.01$

48. $f(x, y, z) = \sqrt{2} \cos x \sin(y + z)$ at $P_0(0, 0, \pi/4)$,
$R: \ |x| \le 0.01, \ |y| \le 0.01, \ |z - \pi/4| \le 0.01$

Estimating Error; Sensitivity to Change

49. Estimating maximum error Suppose that T is to be found from the formula $T = x(e^y + e^{-y})$, where x and y are found to be 2 and $\ln 2$ with maximum possible errors of $|dx| = 0.1$ and $|dy| = 0.02$. Estimate the maximum possible error in the computed value of T.

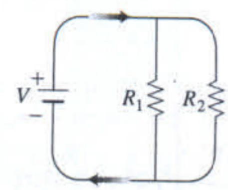
50. Variation in electrical resistance The resistance R produced by wiring resistors of R_1 and R_2 ohms in parallel (see accompanying figure) can be calculated from the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

a. Show that

$$dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2.$$

b. You have designed a two-resistor circuit, like the one shown, to have resistances of $R_1 = 100$ ohms and $R_2 = 400$ ohms, but there is always some variation in manufacturing and the resistors received by your firm will probably not have these exact values. Will the value of R be more sensitive to variation in R_1 or to variation in R_2? Give reasons for your answer.



c. In another circuit like the one shown, you plan to change R_1 from 20 to 20.1 ohms and R_2 from 25 to 24.9 ohms. By about what percentage will this change R?

51. You plan to calculate the area of a long, thin rectangle from measurements of its length and width. Which dimension should you measure more carefully? Give reasons for your answer.

52. a. Around the point $(1, 0)$, is $f(x, y) = x^2(y + 1)$ more sensitive to changes in x or to changes in y? Give reasons for your answer.
b. What ratio of dx to dy will make df equal zero at $(1, 0)$?

53. Value of a 2 × 2 determinant If $|a|$ is much greater than $|b|$, $|c|$, and $|d|$, to which of a, b, c, and d is the value of the determinant

$$f(a, b, c, d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

most sensitive? Give reasons for your answer.

54. The Wilson lot size formula The Wilson lot size formula in economics says that the most economical quantity Q of goods (radios, shoes, brooms, whatever) for a store to order is given by the formula $Q = \sqrt{2KM/h}$, where K is the cost of placing the order, M is the number of items sold per week, and h is the weekly holding cost for each item (cost of space, utilities, security, and so on). To which of the variables K, M, and h is Q most sensitive near the point $(K_0, M_0, h_0) = (2, 20, 0.05)$? Give reasons for your answer.

Theory and Examples

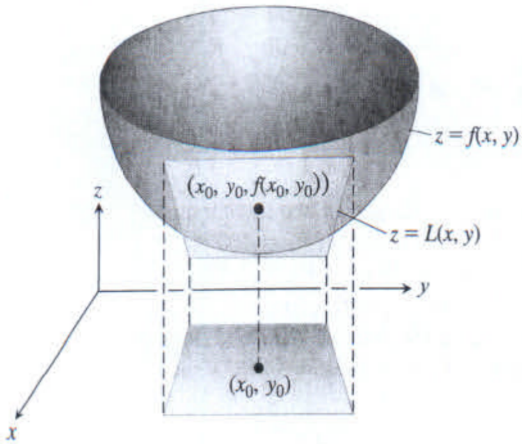
55. The linearization of $f(x, y)$ is a tangent-plane approximation Show that the tangent plane at the point $P_0(x_0, y_0, f(x_0, y_0))$ on the surface $z = f(x, y)$ defined by a differentiable function f is the plane

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

or

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Thus, the tangent plane at P_0 is the graph of the linearization of f at P_0 (see accompanying figure).



56. **Change along the involute of a circle** Find the derivative of $f(x, y) = x^2 + y^2$ in the direction of the unit tangent vector of the curve

$$\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \qquad t > 0.$$

57. **Tangent curves** A smooth curve is *tangent* to the surface at a point of intersection if its velocity vector is orthogonal to ∇f there.

Show that the curve

$$\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} + (2t - 1)\mathbf{k}$$

is tangent to the surface $x^2 + y^2 - z = 1$ when $t = 1$.

58. **Normal curves** A smooth curve is *normal* to a surface $f(x, y, z) = c$ at a point of intersection if the curve's velocity vector is a nonzero scalar multiple of ∇f at the point.

Show that the curve

$$\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - \frac{1}{4}(t + 3)\mathbf{k}$$

is normal to the surface $x^2 + y^2 - z = 3$ when $t = 1$.

14.7 Extreme Values and Saddle Points

Continuous functions of two variables assume extreme values on closed, bounded domains (see Figures 14.41 and 14.42). We see in this section that we can narrow the search for these extreme values by examining the functions' first partial derivatives. A function of two variables can assume extreme values only at domain boundary points or at interior domain points where both first partial derivatives are zero or where one or both of the first partial derivatives fail to exist. However, the vanishing of derivatives at an interior point (a, b) does not always signal the presence of an extreme value. The surface that is the graph of the function might be shaped like a saddle right above (a, b) and cross its tangent plane there.

HISTORICAL BIOGRAPHY
Siméon-Denis Poisson
(1781–1840)

Derivative Tests for Local Extreme Values

To find the local extreme values of a function of a single variable, we look for points where the graph has a horizontal tangent line. At such points, we then look for local maxima, local minima, and points of inflection. For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a horizontal tangent *plane*. At such points, we then look for local maxima, local minima, and saddle points. We begin by defining maxima and minima.

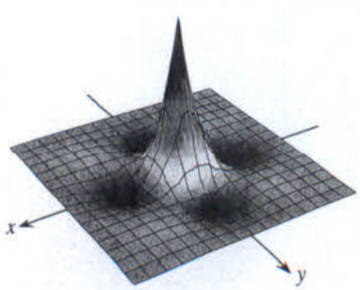


FIGURE 14.41 The function

$$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$$

has a maximum value of 1 and a minimum value of about -0.067 on the square region $|x| \leq 3\pi/2$, $|y| \leq 3\pi/2$.

DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

Local maxima correspond to mountain peaks on the surface $z = f(x, y)$ and local minima correspond to valley bottoms (Figure 14.43). At such points the tangent planes, when they exist, are horizontal. Local extrema are also called **relative extrema**.