

Math 21C
 Kouba
 Discussion Sheet 8

1.) Evaluate the following limits or determine that the limit does not exist.

$$\begin{array}{ll}
 \text{a.) } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2 - 4}{x + y + 2} & \text{b.) } \lim_{(x,y) \rightarrow (1,1)} \frac{xy - y - 2x + 2}{x - 1} \\
 \text{c.) } \lim_{(x,y) \rightarrow (2,2)} \frac{x + y - 4}{\sqrt{x + y} - 2} & \text{d.) } \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} \\
 \text{e.) } \lim_{(x,y) \rightarrow (1,1)} \frac{\sin(x^2 - y^2)}{x - y} & \text{f.) } \lim_{(x,y) \rightarrow (1,-1)} \arcsin \frac{xy}{\sqrt{x^2 + y^2}} \\
 \text{g.) } \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^3 + y^3} & \text{h.) } \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} & \text{i.) } \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} \\
 \text{j.) } \lim_{(x,y) \rightarrow (2,-2)} \frac{4 - xy}{4 + xy} & \text{k.) } \lim_{(x,y) \rightarrow (0,0)} (1 + 3xy^2)^{2/xy^2} & \text{l.) } \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} \\
 \text{m.) } \lim_{(x,y) \rightarrow (1,-2)} \frac{(x-1)^2 + 3(y+2)^2}{x-1 + (y+2)^2} & \text{n.) } \lim_{(x,y) \rightarrow (1,2)} \frac{xy + 2x - y - 2}{xy - y + 3x - 3}
 \end{array}$$

2.) Compute z_x and z_y for each of the following functions.

$$\begin{array}{lll}
 \text{a.) } z = xy^2 + \ln x + e^y + 5 & \text{b.) } z = xe^{2y} \arctan x & \text{c.) } z = \sqrt{x - y^2} \\
 \text{d.) } z = \frac{x^3}{y^2} + \sin(xy) & \text{e.) } z = \frac{x + 4}{x^2 + y^2} & \text{f.) } z = \{e^{x^2y} + \tan(3y + 4x)\}^5 \\
 \text{f.) } z = y^{1+x^3} & &
 \end{array}$$

3.) Show that $z = \ln(1 + x^2 + y^2)$ satisfies the equation $z_{xy} + z_x z_y = 0$.

4.) Verify that $w_{xy} = w_{yx}$ for $w = y + \frac{x}{y}$.

5.) Determine functions z whose partial derivatives are given, or state that this is impossible.

$$\begin{array}{ll}
 \text{a.) } z_x = 2x \text{ and } z_y = 3y^2 + 1 & \text{b.) } z_x = xy^2 - y \text{ and } z_y = x^2y - x \\
 \text{c.) } z_x = e^x y - 1 \text{ and } z_y = e^x - x & \\
 \text{d.) } z_x = ye^x \cos(xy) + e^x \sin(xy) - 2 \text{ and } z_y = xe^x \cos(xy) + 1 &
 \end{array}$$

6.) Consider the function $f(x, y) = \begin{cases} \frac{\sin(x^3 + y)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$.

- Determine $f_x(x, y)$ when $(x, y) \neq (0, 0)$.
- Determine $f_x(0, 0)$ (Use limit definition of partial derivative).
- Determine $f_y(0, 0)$ (Use limit definition of partial derivative).

7.) Plane A, parallel to the xz -plane, and plane B, parallel to the yz -plane, pass through the surface determined by the equation $z = xy^2 - x^3 + 7$. Both planes include the point

$(1, 0, 6)$, which lies on the surface.

a.) Determine the slope of the line tangent to the surface at the point $(1, 0, 6)$ if the line lies in

i.) plane A.

ii.) plane B.

b.) Determine an equation of the plane tangent to the surface at the point $(1, 0, 6)$.

8.) Compute z_x and z_y for each of the following functions.

a.) $z = x^3y + y^4 - 2x + 5$ b.) $z = f(x) + g(y)$ c.) $z = f(x^3) + g(4y)$

d.) $z = f(x^2 + y^3) + g(xy^2)$ e.) $y^2 + z^2 + \sin(xz) = 4$

f.) $z = f(u, v)$ where $u = \ln(x - y)$ and $v = e^{xy}$

9.) Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$ if $w = f(4t^2 - 3s)$ and $f'(x) = \ln x$.

10.) Assume that f is differentiable function of one variable with $z = xf(xy)$. Show that $xz_x - yz_y = z$.

11.) Assume that f and g are twice differentiable functions of one variable. Show that

$u = f(x + at) + g(x - at)$ satisfies $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, where a is a constant.

12.) Consider the paraboloid given by $f(x, y) = 25 - x^2 - y^2$.

a.) Sketch the surface.

b.) Let point $P = (2, -2)$. Compute the derivative of the function f at the point P in the direction

i.) $\vec{A} = \overrightarrow{(-3, 4)}$

ii.) $\vec{A} = \overrightarrow{(3, -4)}$

iii.) $\vec{A} = \overrightarrow{(1, 0)}$

iv.) $\vec{A} = \overrightarrow{(0, -1)}$

c.) In what directions is the derivative of f at point $P = (2, -2)$ equal to zero ?

d.) In what directions is the derivative of f at point $P = (-1, 1)$ equal to 2 ?

THE FOLLOWING PROBLEM IS FOR RECREATIONAL PURPOSES ONLY.

13.) Determine the exact value of the "continued" square root :

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$$