Math 21C
Koulba
The Limit of Sequence $\left\{ \frac{x^n}{n!} \right\}$

<u>Example</u> : Find the limit of the sequence $\left\{ \frac{3^n}{n!} \right\}$. Start by writing out the first six terms :

$$n: \quad 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6$$

$$\frac{3^n}{n!} : \quad \frac{3}{1}, \frac{3\cdot 3}{2\cdot 1}, \frac{3\cdot 3\cdot 3}{3\cdot 2\cdot 1}, \frac{3\cdot 3\cdot 3\cdot 3}{4\cdot 3\cdot 2\cdot 1}, \frac{3\cdot 3\cdot 3\cdot 3\cdot 3}{5\cdot 4\cdot 3\cdot 2\cdot 1}, \frac{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}, \dots$$

$$= 3, \frac{9}{2}, \frac{9}{2}, \left(\frac{3}{4}\right)\left(\frac{9}{2}\right), \left(\frac{3}{5}\right)\left(\frac{3}{4}\right)\left(\frac{9}{2}\right), \left(\frac{3}{6}\right)\left(\frac{3}{5}\right)\left(\frac{3}{4}\right)\left(\frac{9}{2}\right), \dots$$

$$\leq 3, \frac{9}{2}, \left(\frac{3}{4}\right)^0\left(\frac{9}{2}\right), \left(\frac{3}{4}\right)^1\left(\frac{9}{2}\right), \left(\frac{3}{4}\right)^2\left(\frac{9}{2}\right), \left(\frac{3}{4}\right)^3\left(\frac{9}{2}\right), \dots$$

Thus, for $n \geq 3$ it follows that

$$0 \leq \frac{3^n}{n!} \leq \left(\frac{3}{4}\right)^{n-3}\left(\frac{9}{2}\right).$$

Since $\lim_{n\to\infty} \left(\frac{3}{4}\right)^{n-3}\left(\frac{9}{2}\right) = (0)\left(\frac{9}{2}\right) = 0$, it follows from the Sandwich Theorem that $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

<u>Rule</u> : Let x be any real number. Then

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$