

Math 21C
Kouba
Geometric Series Test

Let r be a real number and consider the geometric series given by

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \dots .$$

If $r = 1$, then $\sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + 1 + \dots$ diverges ($= \infty$) by the sequence of partial sums

test. If $r = -1$, then $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$ diverges (oscillation) by the sequence of partial sums test. What happens with other values of r ? Apply the sequence of partial sums test :

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 + r, \\ s_3 &= 1 + r + r^2, \\ s_4 &= 1 + r + r^2 + r^3, \dots \end{aligned}$$

and

$$s_n = 1 + r + r^2 + r^3 + \dots + r^{n-2} + r^{n-1} = \frac{1 - r^{(n-1)+1}}{1 - r} = \frac{1 - r^n}{1 - r} .$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} r^n &= \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1 - r^n}{1 - r} \\ &= \begin{cases} \frac{1 - 0}{1 - r} = \frac{1}{1 - r} , & \text{if } -1 < r < 1 \\ \text{Does Not Exist} , & \text{if } r > 1 \text{ or } r < -1 . \end{cases} \end{aligned}$$

Thus, the geometric series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \dots = \frac{1}{1 - r}$$

for $-1 < r < 1$. This series diverges for all other values of r .