Math 21C Midterm I Friday, April 19 Spring 2024

Name:

Student ID:

You may not use a calculator.
You may use one page of notes.
You may not use the textbook.
Please do not simplify answers.

1. (9 pts each: Series)
   Determine for each part whether the series converges or diverges.
   Write clear and complete solutions including the name of the series test you use and what your answer is.

   (a)

$$\sum_{n=1}^{\infty} \frac{3n^2+1}{n^2+n}$$

$$\lim_{n\to\infty} \frac{3n^2+1}{n^2+n} = 3, \text{ so } \sum_{n=1}^{\infty} \frac{3n^2+1}{n^2+n}$$

diverges by nth term test

   (b)

$$\sum_{n=1}^{\infty} \left(\frac{-3}{2}\right)^n$$

$$\left|\frac{-3}{2}\right| = \frac{3}{2} > 1, \text{ so } \sum_{n=1}^{\infty} \left(\frac{-3}{2}\right)^n \text{ diverges}$$

by geometric series test.

(c)

$$\sum_{n=1}^{\infty}\sqrt{\frac{n+1}{n^3+5}} \approx \sum_{n=1}^{\infty}\frac{1}{n}$$

Using LCT,

$$\lim_{n\to\infty}\frac{\sqrt{\frac{n+1}{n^3+5}}}{\frac{1}{n}} = \lim_{n\to\infty}\sqrt{\frac{n+1}{n^3+5}}\sqrt{n^2} = \lim_{n\to\infty}\sqrt{\frac{n^3+n^2}{n^3+5}} = 1$$

$1<\infty$, so by LCT as $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent p-series,

then $\sum_{n=1}^{\infty}\sqrt{\frac{n+1}{n^3+5}}$ diverges.

(d)

$$\sum_{n=1}^{\infty}(-1)^n\sqrt{\frac{n+1}{n^3+5}}$$

<u>Alternating Series Test:</u>

Positivity: $\sqrt{\frac{n+1}{n^3+5}} > 0$ for all $n \geq 1$.

Decreasing: Consider $f(x)=\sqrt{\frac{x+1}{x^3+5}}$. $f'(x)=\frac{1}{2}\left(\frac{x+1}{x^3+5}\right)^{-1/2}\left(\frac{x^3+5-3x^2(x+1)}{(x^3+5)^2}\right)$

$$= \underbrace{\frac{1}{2}\sqrt{\frac{x^3+5}{x+1}}}_{>0}\left(\frac{-2x^3-3x^2+5}{\underbrace{(x^3+5)^2}_{>0}}\right)$$

$-2x^3-3x^2+5<0$ if $x \geq 2$, so decreasing when $n \geq 2$.

Limits to 0:

$$\lim_{n\to\infty}\sqrt{\frac{n+1}{n^3+5}} = 0.$$

Thus, by Alt series test, $\sum_{n=1}^{\infty}(-1)^n\sqrt{\frac{n+1}{n^3+5}}$ converges.

(e) $\sum_{n=1}^{\infty} ne^{-n^2}$

Positivity: ✓

$ne^{-n^2} > 0 \ \forall n$

Decreasing: ✓

$f(x) = xe^{-x^2}$

$f'(x) = e^{-x^2} - 2x^2e^{-x^2}$

$= \underbrace{(1-2x^2)}_{<0 \ \forall x.} e^{-x^2}$

Integral Test:

$u = -x^2, \ du = -2x\,dx$

$\int_1^{\infty} xe^{-x^2}dx = -\frac{1}{2}\int e^{u}du$

$= -\frac{1}{2}e^{-x^2}\Big|_1^{\infty} = \frac{1}{2e}$

As $\int_1^{\infty} xe^{-x^2}dx$ converges, by the integral test, $\sum_{n=1}^{\infty} ne^{-n^2}$ converges.

(f) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

*Note $\sin(n) \not> 0 \ \forall n$.

Absolute convergence

$\frac{-1}{n^2} \le \frac{\sin(n)}{n^2} \le \frac{1}{n^2}$, so

$\left|\frac{\sin(n)}{n^2}\right| \le \frac{1}{n^2}$

Thus, As $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series with $p=2$, $\sum_{n=1}^{\infty}\left|\frac{\sin(n)}{n^2}\right|$ converges. Thus, $\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}$ converges absolutely. (ie it converges)

(g)

$$\sum_{n=1}^{\infty} \left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}} \right)$$

This is a telescoping sum. Thus,

$$\sum_{n=1}^{\infty} e^{1/n} - e^{\frac{1}{n+1}} = e - \lim_{n\to\infty} e^{\frac{1}{n+1}}$$

$$= e - e^{0}$$

$$= e - 1$$

(h)

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$$

Recall $\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k$.

$\Rightarrow \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$, so

$\sum_{n=1}^{\infty} \left(1 - 1/n\right)^n$ diverges by nth term test.

2. (10 pts: Story)
A redwood tree increases in diameter each spring. Each spring its diameter grows 99 percent as much as it did the previous spring. During its first spring its diameter grows to one foot.
What will be the eventual diameter of the tree if it lives forever?

$a=1$, $r=.99$. Let D=diameter.

$$D = \underbrace{\sum_{n=0}^{\infty} 1(.99)^n}_{\text{geometric, } r=.99<1} = \frac{1}{1-.99} = 100$$

The tree will grow to a diameter of 100ft.

3. (9 pts: Integral Errors)
The series

$$T = \sum_{n=1}^{\infty} n e^{-n^2}$$

converges rapidly.

(a) Find any upper and lower bounds for T.

$$\frac{1}{e} \leq \sum_{n=1}^{\infty} n e^{-n^2} \leq \frac{1}{e} + \frac{1}{2e}$$

Proven in part b.

(b) Find upper and lower bounds for T which differ by at most $\frac{1}{2}$.

Recall: $\sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{\infty} f(k) \leq \sum_{k=1}^{n} f(k) + \int_{n}^{\infty} f(x)\,dx$

ie, we want $\int_{n}^{\infty} x e^{-x^2}\,dx \leq 1/2$, solving for n.

$$\int_{n}^{\infty} x e^{-x^2}\,dx = -\frac{1}{2} e^{-x^2} \Big|_{n}^{\infty} = \frac{1}{2e^{n}} \leq 1/2$$

$$\frac{1}{e^n} \leq 1$$

$1 \leq e^n$, so $0 \leq n$, but n starts at 1, so $1 \leq n$.

So,

$$\boxed{\frac{1}{e} \leq \sum_{n=1}^{\infty} n e^{-n^2} \leq \frac{1}{e} + \int_{1}^{\infty} x e^{-x^2}\,dx = \frac{1}{e} + \frac{1}{2e}}$$

4. (9 pts: Alternating Errors)
The alternating series

$$S = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

converges slowly.

(a) Find any upper and lower bounds for S.

$$S_2 = \frac{(-1)^2}{\ln(2)} = \frac{1}{\ln(2)}$$

$$S_3 = \frac{1}{\ln(2)} - \frac{1}{\ln(3)} \qquad \Rightarrow \qquad S_3 \le S \le S_2$$

Observe that for even n,

$S_n \ge S_{n+1}$.

(b) Find upper and lower bounds for S which differ by at most $\frac{1}{2}$.

$|a_{n+1}| \le$ error, so $\left| \frac{(-1)^{n+1}}{\ln(n+1)} \right| \le 1/2$

$$\Rightarrow \frac{1}{\ln(n+1)} \le 1/2 \Rightarrow 2 \le \ln(n+1)$$

$$\Rightarrow e^2 \le n+1$$

$$\Rightarrow e^2 - 1 \le n$$

$$\Rightarrow 3^2 - 1 = 8 \le n.$$

e not natural #, so use $3 > e$.

So,

$$\boxed{\sum_{n=2}^{9} \frac{(-1)^n}{\ln(n)} \le \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} \le \sum_{n=2}^{8} \frac{(-1)^n}{\ln(n)}}$$

5. (10 pts: Extra Credit... you may skip this problem)
You know that $\sum_{m=0}^{\infty} 7r^m = 8$. Find the exact value of

$$\sum_{m=0}^{\infty} 7\sqrt[3]{r^m}.$$

$$\underbrace{\sum_{m=0}^{\infty} 7r^m = 8}_{\text{geometric,}}$$

$$\Rightarrow \frac{7}{1-r} = 8$$

$$\frac{7}{8} = 1-r$$

$$r = 1 - \frac{7}{8} = \frac{1}{8}$$

$$\sum_{m=0}^{\infty} 7\sqrt[3]{r^m} = \sum_{m=0}^{\infty} 7\sqrt[3]{\left(\frac{1}{8}\right)^m}$$

$$= \sum_{m=0}^{\infty} 7\left(\sqrt[3]{\tfrac{1}{8}}\right)^m$$

$$= \sum_{m=0}^{\infty} 7\left(\frac{1}{2}\right)^m$$

$\left|\frac{1}{2}\right| < 1$, so

$$= \frac{7}{1-\frac{1}{2}} = \boxed{14}$$