Math 21C
Kouba
The Sequence of Partial Sums Test

RECALL : A sequence $\{a_n\}$ is a function which assigns a real number a_n to each natural number n : $1, 2, 3, 4, 5, \cdots$, i.e., a sequence is an ordered list of real numbers : $a_1, a_2, a_3, a_4, a_5, \cdots$.

EXAMPLE : $\left\{\frac{2^{n-1}}{n+7}\right\}$ generates the sequence $\frac{1}{8}, \frac{2}{9}, \frac{4}{10}, \frac{8}{11}, \frac{16}{12}, \cdots$.

DEFINITION : An infinite series $\sum_{n=1}^{\infty} a_n$ is the *sum* of the numbers in the sequence $\{a_n\}$, i.e., $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$.

EXAMPLE : $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$.

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But how do we add an infinite number of numbers together ? We need a more precise definition of an infinite series $\sum_{n=1}^{\infty} a_n$. Begin by constructing a new sequence of partial sums by letting (This step by step process will be called the Sequence of Partial Sums Test for the infinite series $\sum_{n=1}^{\infty} a_n$.)

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ s_4 &= a_1 + a_2 + a_3 + a_4, \\ &\\ s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n \ . \end{aligned}$$

We can now say that the value of the infinite series is precisely the value of the limit of its sequence of partial sums, i.e.,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= a_1 + a_2 + a_3 + a_4 + \cdots \\ &= \lim_{n \to \infty} (a_1 + a_2 + a_3 + a_4 + \cdots + a_n) \\ &= \lim_{n \to \infty} s_n \ . \end{aligned}$$