Math 21C
Kouba
The Ratio Test and Root Test

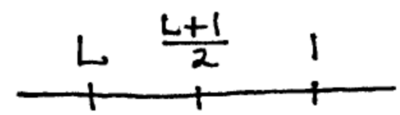
<u>Ratio Test</u>: assume $a_n > 0$ and consider $\sum_{n=1}^{\infty} a_n$.

1.) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

2.) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

3.) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then the ratio test is inconclusive.

<u>Proof</u>: 1.) assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$.



Then there is some integer N so that $\frac{a_{n+1}}{a_n} < \frac{L+1}{2} = m$ for all $n \geq N$. Thus, $a_{n+1} < m\, a_n$ for all $n \geq N$, so that

$$a_{N+1} < m\, a_N ,$$
$$a_{N+2} < m\, a_{N+1} < m^2 a_N ,$$
$$a_{N+3} < m\, a_{N+2} < m^3 a_N ,$$
$$a_{N+4} < m\, a_{N+3} < m^4 a_N , \ \cdots .$$

It follows that

$$\begin{aligned}
\sum_{n=N}^{\infty} a_n &= a_N + a_{N+1} + a_{N+2} + a_{N+3} + \cdots \\
&\leq a_N + m a_N + m^2 a_N + m^3 a_N + \cdots \\
&= a_N (1 + m + m^2 + m^3 + \cdots) \\
&= a_N \cdot \frac{1}{1-m} \qquad (\text{since } -1 < m < 1) \\
&< \infty .
\end{aligned}$$

This means that $\sum_{n=N}^{\infty} a_n$ converges, so that $\sum_{n=1}^{\infty} a_n$ converges.

2.) Assume $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L > 1$. Then there is some integer N so that $\frac{a_{n+1}}{a_n} > 1$ for all $n \geq N$. Thus, $a_{n+1} > a_n$ for all $n \geq N$, so that

$$0 < a_N < a_{N+1} < a_{N+2} < a_{N+3} \cdots .$$

This means that a_n is a positive, increasing sequence for $n \geq N$. It follows that $\lim_{n \to \infty} a_n \neq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges by the nth term test.

<u>Root Test</u>: Assume $a_n > 0$ and consider $\sum_{n=1}^{\infty} a_n$.

1.) If $\lim_{n \to \infty} a_n^{1/n} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

2.) If $\lim_{n \to \infty} a_n^{1/n} = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

3.) If $\lim_{n \to \infty} a_n^{1/n} = 1$, then the root test is inconclusive.

<u>Proof</u>: Similar to Ratio Test Proof.