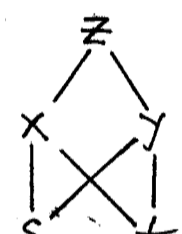
Math 21C
Kouba
Finding the Second Partial Derivative Using the Chain Rule

Assume that we are given the functions $z = f(x, y)$, $x = g(s, t)$, and $y = k(s, t)$. Our goal is to determine the form of the second partial derivative of z with respect to t, $\frac{\partial^2 z}{\partial t^2}$. (In a similar fashion we can determine $\frac{\partial^2 z}{\partial s^2}$.) We will use the diagrams on the right to guide us. The first partial derivative of z with respect to t is

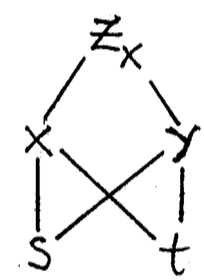
$$\frac{\partial z}{\partial t} = z_x \cdot \frac{\partial x}{\partial t} + z_y \cdot \frac{\partial y}{\partial t} .$$



The second partial derivative is now

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial t}\left(z_x \cdot \frac{\partial x}{\partial t} + z_y \cdot \frac{\partial y}{\partial t}\right)$$

(Use the Product Rule twice and again use the Chain Rule twice.)

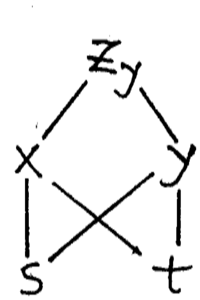


$$= \left\{z_x \cdot \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}(z_x) \cdot \frac{\partial x}{\partial t}\right\}$$

$$+ \left\{z_y \cdot \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial}{\partial t}(z_y) \cdot \frac{\partial y}{\partial t}\right\}$$

$$= z_x \cdot \frac{\partial^2 x}{\partial t^2} + \left[z_{xx} \cdot \frac{\partial x}{\partial t} + z_{xy} \cdot \frac{\partial y}{\partial t}\right] \cdot \frac{\partial x}{\partial t}$$

$$+ z_y \cdot \frac{\partial^2 y}{\partial t^2} + \left[z_{yx} \cdot \frac{\partial x}{\partial t} + z_{yy} \cdot \frac{\partial y}{\partial t}\right] \cdot \frac{\partial y}{\partial t}$$



$$= z_x \cdot \frac{\partial^2 x}{\partial t^2} + z_y \cdot \frac{\partial^2 y}{\partial t^2} + z_{xx} \cdot \left(\frac{\partial x}{\partial t}\right)^2$$

$$+ 2z_{xy} \cdot \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial t}\right) + z_{yy} \cdot \left(\frac{\partial y}{\partial t}\right)^2 .$$