Math 21C
Kouba
Problems Using (*) and (*)(*)

1.) a.) $\int_1^{11} \frac{1}{x}\,dx < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10} < 1+\int_1^{10}\frac{1}{x}\,dx \rightarrow$

$\ln x\Big|_1^{11} < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10} < 1+\ln x\Big|_1^{10} \rightarrow$

$\ln 11-\cancel{\ln 1}^{\,0} < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10} < 1+\ln 10-\cancel{\ln 1}^{\,0} \rightarrow$

$2.398 \approx \ln 11 < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10} < 1+\ln 10 \approx 3.303$

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b.) $\int_1^{1001} \frac{1}{x}\,dx < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1000} < 1+\int_1^{1000}\frac{1}{x}\,dx \rightarrow$

$\int_1^{1001}\frac{1}{x}\,dx = \ln x\Big|_1^{1001} = \ln 1001-\cancel{\ln 1}^{\,0} = \ln 1001$ and

$\int_1^{1000}\frac{1}{x}\,dx = \ln x\Big|_1^{1000} = \ln 1000-\cancel{\ln 1}^{\,0} = \ln 1000$ so

$6.909 \approx \ln 1001 < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1000} < 1+\ln 1000 \approx 7.908$

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c.) $\int_1^{1{,}000{,}001} \frac{1}{x}\,dx < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1{,}000{,}000} < 1+\int_1^{1{,}000{,}000}\frac{1}{x}\,dx \rightarrow$

$13.816 \approx \ln 1{,}000{,}001 < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1{,}000{,}000} < 1+\ln 1{,}000{,}000 \approx 14.816$

2.) a.) $S_{10} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{10^2} \approx 1.550$

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b.) $\int_{11}^{\infty}\frac{1}{x^2}\,dx < \frac{1}{11^2}+\frac{1}{12^2}+\cdots < \int_{10}^{\infty}\frac{1}{x^2}\,dx \rightarrow$

$\lim_{A\to\infty}\int_{11}^{A}\frac{1}{x^2}\,dx = \lim_{A\to\infty}\frac{-1}{x}\Big|_{11}^{A} = \lim_{A\to\infty}\left(\cancel{\frac{-1}{A}}^{\,0}-\frac{-1}{11}\right) = \frac{1}{11}$ and

$\lim_{A\to\infty}\int_{10}^{A}\frac{1}{x^2}\,dx = \lim_{A\to\infty}\frac{-1}{x}\Big|_{10}^{A} = \lim_{A\to\infty}\left(\frac{-1}{A}-\frac{-1}{10}\right) = \frac{1}{10}$ so

$$0.091 \approx \frac{1}{11} < \frac{1}{11^2} + \frac{1}{12^2} + \cdots < \frac{1}{10} = 0.1$$

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c.) $$\int_{101}^{\infty} \frac{1}{x^2}\,dx < R_{100} < \int_{100}^{\infty} \frac{1}{x^2}\,dx \rightarrow$$

$$\lim_{A\to\infty} \int_{101}^{A} \frac{1}{x^2}\,dx = \lim_{A\to\infty} \frac{-1}{x}\Big|_{101}^{A} = \lim_{A\to\infty}\left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{101}\right) = \frac{1}{101} \text{ and}$$

$$\lim_{A\to\infty} \int_{100}^{A} \frac{1}{x^2}\,dx = \lim_{A\to\infty} \frac{-1}{x}\Big|_{100}^{A} = \lim_{A\to\infty}\left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{100}\right) = \frac{1}{100} \text{ so}$$

$$0.0099 \approx \frac{1}{101} < R_{100} < \frac{1}{100} = 0.01$$

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d.) $$R_n < \int_n^{\infty} \frac{1}{x^2}\,dx \le 0.0001 \rightarrow$$

$$\lim_{A\to\infty} \int_n^{A} \frac{1}{x^2}\,dx \le 0.0001 \rightarrow \lim_{A\to\infty} \frac{-1}{x}\Big|_n^{A} \le 0.0001$$

$$\rightarrow \lim_{A\to\infty}\left(\cancelto{0}{\frac{-1}{A}} - \frac{-1}{n}\right) \le 0.0001 \rightarrow \frac{1}{n} \le 0.0001 \rightarrow$$

$$n \ge 10{,}000.$$

3.) a.) $$S_{10} = 1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^9 = \frac{1-\left(\frac{2}{3}\right)^{10}}{1-\left(\frac{2}{3}\right)} \approx 2.9480$$

b.) $$\int_{11}^{\infty} \left(\frac{2}{3}\right)^{x-1} dx < R_{10} < \int_{10}^{\infty} \left(\frac{2}{3}\right)^{x-1} dx \rightarrow$$

$$\lim_{A\to\infty} \int_{11}^{A} \left(\frac{2}{3}\right)^{x-1} dx = \lim_{A\to\infty} \frac{\left(\frac{2}{3}\right)^{x-1}}{\ln(2/3)}\Bigg|_{11}^{A}$$

$$= \lim_{A\to\infty}\left(\cancelto{0}{\frac{\left(\frac{2}{3}\right)^{A-1}}{\ln(2/3)}} - \frac{\left(\frac{2}{3}\right)^{10}}{\ln(2/3)}\right) = \frac{-\left(\frac{2}{3}\right)^{10}}{\ln(2/3)} \approx 0.0427 \text{ and}$$

$$\lim_{A\to\infty} \int_{10}^{A} \left(\frac{2}{3}\right)^{x-1} dx = \lim_{A\to\infty} \frac{\left(\frac{2}{3}\right)^{x-1}}{\ln(2/3)}\Bigg|_{10}^{A}$$

$$= \lim_{A\to\infty}\left(\frac{\cancel{\left(\frac{2}{3}\right)^{A-1}}^{\,0}}{\ln(2/3)} - \frac{\left(\frac{2}{3}\right)^{9}}{\ln(2/3)}\right) = \frac{-\left(\frac{2}{3}\right)^{9}}{\ln(2/3)} \approx 0.0642 \text{ so}$$

$$0.0427 < R_{10} < 0.0642 .$$

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c.) $R_{10} = \left(\frac{2}{3}\right)^{10} + \left(\frac{2}{3}\right)^{11} + \left(\frac{2}{3}\right)^{12} + \cdots = \left(\frac{2}{3}\right)^{10}\cdot\left[1+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\cdots\right]$

$$= \left(\frac{2}{3}\right)^{10}\cdot\frac{1}{1-\left(\frac{2}{3}\right)} \approx 0.0520$$

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d.) $R_n < \int_n^\infty \left(\frac{2}{3}\right)^{x-1} dx \le 0.0001 \rightarrow$

$\lim_{A\to\infty} \int_n^A \left(\frac{2}{3}\right)^{x-1} dx \le 0.0001 \rightarrow$

$\lim_{A\to\infty} \left.\frac{\left(\frac{2}{3}\right)^{x-1}}{\ln(2/3)}\right|_n^A = \lim_{A\to\infty}\left(\frac{\cancel{\left(\frac{2}{3}\right)^{A-1}}^{\,0}}{\ln(2/3)} - \frac{\left(\frac{2}{3}\right)^{n-1}}{\ln(2/3)}\right) \le 0.0001 \rightarrow$

$\frac{-\left(\frac{2}{3}\right)^{n-1}}{\ln(2/3)} \le 0.0001 \rightarrow \left(\frac{2}{3}\right)^{n-1} \le -(0.0001)\ln(2/3) \rightarrow$

$(n-1)\ln\left(\frac{2}{3}\right) \le \ln[-(0.0001)\ln(2/3)] \rightarrow$

$n-1 \ge \frac{\ln[-(0.0001)\ln(2/3)]}{\ln(2/3)} \rightarrow$

$n \ge \frac{\ln[-(0.0001)\ln(2/3)]}{\ln(2/3)} + 1 \approx 25.9$ so

choose $n \ge 26$.

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e.) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots$

$$= \frac{1}{1-\left(\frac{2}{3}\right)} = 3 .$$