Math 21C (Spring 2006) PRACTICE EXAM
Kouba
Exam 1

Please PRINT your name here : KEY

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Your Exam ID Number ________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. COPYING ANSWERS FROM ANOTHER STUDENT'S EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 5:00 p.m. sharp to finish the exam. PLEASE STOP WRITING IMMEDIATELY when time is called and close your exam.

7. Make sure that you have 5 pages including the cover page.

8. The following may be used on this exam.

$$(*) \int_1^{n+1} f(x)\,dx < f(1) + f(2) + \cdots + f(n) < f(1) + \int_1^n f(x)\,dx$$

$$(*)(*) \int_{n+1}^{\infty} f(x)\,dx < f(n+1) + f(n+2) + f(n+3) + \cdots < \int_n^{\infty} f(x)\,dx$$

1.) (9 pts. each) Determine whether each of the following series converges or diverges. Write clear and complete solutions including the name of the series test that you use and what your final answer is.

a.) $\sum_{n=1}^{\infty} \frac{n+2}{3n+5}$; $\lim_{n\to\infty} \frac{n+2}{3n+5} = \lim_{n\to\infty} \frac{1+2/n}{3+5/n}$

$= \frac{1+0}{3+0} = \frac{1}{3} \neq 0$ so series diverges by nth-term test

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b.) $\sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n$ converges by geometric series test since $r = -\frac{2}{3}$, $-1 < r < 1$.

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c.) $\sum_{n=2}^{\infty} \sqrt{\frac{n}{n^4+3}}$; $\lim_{n\to\infty} \frac{\sqrt{\frac{n}{n^4+3}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \sqrt{\frac{n}{n^4+3}} \cdot \frac{\sqrt{n^3}}{1}$

$= \lim_{n\to\infty} \sqrt{\frac{n^4}{n^4+3}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+3/n^4}} = \sqrt{\frac{1}{1+0}} = 1$, so series converges since $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series ($p = 3/2 > 1$) by limit comparison test.

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d.) $\sum_{n=3}^{\infty} \frac{(\ln n)^2}{n}$ (Use the Integral Test.)

Let $f(x) = \frac{(\ln x)^2}{x} \xrightarrow{D} f'(x) = \frac{\cancel{x} \cdot 2\ln x \cdot \frac{1}{\cancel{x}} - (\ln x)^2}{x^2}$

$= \frac{\ln x \cdot (2 - \ln x)}{x^2} \rightarrow \quad + \quad \underset{e^2 \approx 7.4}{\overset{0}{|}} \quad - \qquad f'$

so f is +, ↓, and continuous for $x \geq 8$. then $\int_8^{\infty} \frac{(\ln x)^2}{x}\,dx = \lim_{A\to\infty} \frac{1}{3}(\ln x)^3 \Big|_8^A = \lim_{A\to\infty} \left(\frac{1}{3}(\ln A)^3 - \frac{1}{3}(\ln 8)^3\right)$

$= \infty$ so series $\sum_{n=8}^{\infty} \frac{(\ln n)^2}{n}$ diverges by integral test and $\sum_{n=3}^{\infty} \frac{(\ln n)^2}{n}$ diverges by "subtle facts".

e.) $\sum_{n=1}^{\infty} \frac{5^{n+1}}{(2n)!}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+2}}{(2(n+1))!} \cdot \frac{(2n)!}{5^{n+1}}$

$= \lim_{n\to\infty} \frac{5}{(2n+2)(2n+1)} = \text{"}\frac{5}{\infty}\text{"} = 0 < 1$, so series converges by ratio test.

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f.) $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{n+\sqrt{n}}$; Let $a_n = \frac{1}{n+\sqrt{n}}$, then a_n is $+, \downarrow$, and $\lim_{n\to\infty} a_n = 0$, so series converges by alternating series test.

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g.) $\frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} - \frac{1}{9^4} + \cdots$

The series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by p-series test ($p = 4 > 1$), so given series converges by the absolute convergence test

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h.) $\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ (Use the Sequence of Partial Sums Test.)

$S_1 = \frac{1}{2} - \frac{1}{\sqrt{5}}$,

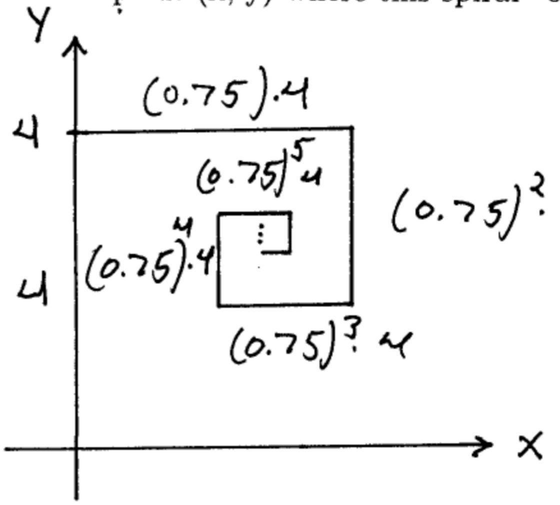
$S_2 = \left(\frac{1}{2} - \cancel{\frac{1}{\sqrt{5}}}\right) + \left(\cancel{\frac{1}{\sqrt{5}}} - \frac{1}{\sqrt{6}}\right) = \frac{1}{2} - \frac{1}{\sqrt{6}}$,

$S_3 = \left(\frac{1}{2} - \cancel{\frac{1}{\sqrt{5}}}\right) + \left(\cancel{\frac{1}{\sqrt{5}}} - \cancel{\frac{1}{\sqrt{6}}}\right) + \left(\cancel{\frac{1}{\sqrt{6}}} - \frac{1}{\sqrt{7}}\right) = \frac{1}{2} - \frac{1}{\sqrt{7}}$,

$\ldots S_n = \frac{1}{2} - \frac{1}{\sqrt{n+4}}$; then

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{\sqrt{n+4}}\right) = \frac{1}{2} - 0 = \frac{1}{2}$ → so series converges by sequence of partial sums test.

2.) (10 pts.) Start at the origin and move 4 units along the positive y-axis. Turn 90 degrees to the right and move 75% of that distance. Turn 90 degrees to the right and move 75% of that distance. Turn 90 degrees to the right and move 75% of that distance. Continue this process forming a "spiral with square corners." Determine the *y-coordinate* for the point (x, y) where this spiral "ends."



y-value is

$$y = 4 - (0.75)^2\cdot 4 + (0.75)^4\cdot 4 - (0.75)^6\cdot 4 + (0.75)^8\cdot 4 - \cdots$$

$$= 4\left[1 + \left(-(0.75)^2\right) + \left(-(0.75)^2\right)^2 + \left(-(0.75)^2\right)^3 + \cdots\right.$$

$$= 4\left[1 + \left(-\frac{9}{16}\right) + \left(-\frac{9}{16}\right)^2 + \left(-\frac{9}{16}\right)^3 + \cdots\right]$$

$$= 4\cdot\frac{1}{1-\left(-9/16\right)} = 4\cdot\frac{1}{25/16} = 4\cdot\frac{16}{25} = \frac{64}{25} = 2.56$$

3.) (8 pts.) The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+7}}$ converges. What should n be so that the partial sum $S_n = \sum_{i=1}^{n}(-1)^{i+1}\frac{1}{\sqrt{i+7}}$ estimates the exact value of the series with absolute error at most 0.001 ?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{\sqrt{n+7}} = \underbrace{\frac{1}{\sqrt{8}} - \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{10}} - \cdots + (-1)^{n+1}\cdot\frac{1}{\sqrt{n+7}}}_{S_n} + \underbrace{(-1)^{n+2}\cdot\frac{1}{\sqrt{n+8}} + \cdots}_{R_n}$$

$$|R_n| < \frac{1}{\sqrt{n+8}} \leq 0.001 \rightarrow$$

$$\frac{1}{0.001} \leq \sqrt{n+8} \rightarrow 1000 \leq \sqrt{n+8} \rightarrow 1000^2 \leq n+8$$

$$\rightarrow n \geq 1000^2 - 8 \rightarrow \text{choose } n \geq 999{,}992$$

4.) (10 pts.) The series $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)^2}$ converges. What should n be so that the partial sum $S_n = \sum_{i=1}^{n} \frac{1}{i(1+\ln i)^2}$ estimates the exact value of the series with error at most 0.1 ?

$$\text{use } (*)(*) : \underbrace{f(n+1)+f(n+2)+\cdots}_{R_n} < \underline{\underline{\int_n^\infty f(x)\,dx \le 0.1}}$$

$$\rightarrow \int_n^\infty \frac{1}{x(1+\ln x)^2}\,dx = \lim_{A\to\infty} \left.\frac{-1}{1+\ln x}\right|_n^A$$

$$= \lim_{A\to\infty}\left(\frac{-1}{1+\ln A} - \frac{-1}{1+\ln n}\right) = 0 + \frac{1}{1+\ln n} \le 0.1 \rightarrow$$

$$\frac{1}{0.1} \le 1+\ln n \rightarrow 10 \le 1+\ln n \rightarrow \ln n \ge 9 \rightarrow$$

$$e^{\ln n} \ge e^9 \rightarrow \text{choose } n \ge e^9 \approx 8104$$

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) The following series converges. Determine the *exact* value of this infinite series :

$$\frac{1}{1}+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\frac{5}{2^4}+\frac{6}{2^5}+\cdots$$

$$= 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$
$$+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$
$$+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$
$$+\frac{1}{2^3}+\cdots$$

$$\frac{1}{1-\frac{1}{2}} = 2!$$

$$= \left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)+\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)$$
$$+\frac{1}{2^2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)+\frac{1}{2^3}\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)+\cdots$$

$$= 2\cdot\left[1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right]$$

$$= 2\cdot 2 = 4.$$