Math 21C (Spring 2006) PRACTICE EXAM
Kouba
Exam 2

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. COPYING ANSWERS FROM ANOTHER STUDENT'S EXAM IS A VIOLATION OF THE UNIVERSITY HONOR CODE. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 5:00 p.m. sharp to finish the exam. PLEASE STOP WRITING IMMEDIATELY when time is called and close your exam.

7. Make sure that you have 7 pages including the cover page.

1.) (10 pts.) Determine the interval of convergence (including endpoints) for the following power series.

$$\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n}(x-1)^n \quad ; \quad \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{\frac{2^{n+1}}{n+1}|x-1|^{n+1}}{\frac{2^n}{n}|x-1|^n}$$

$$= \lim_{n\to\infty} 2\cdot\frac{n}{n+1}\cdot|x-1| = 2(1)|x-1| = 2|x-1| < 1 \rightarrow$$

$$|x-1| < \frac{1}{2} \rightarrow \frac{-1}{2} < x-1 < \frac{1}{2} \rightarrow \frac{1}{2} < x < \frac{3}{2} \quad ;$$

check $x = \frac{3}{2}$: $\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}$ is a convergent alternating series since $a_n = \frac{1}{n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$.

$(-1)^{2n} = (-1)^{2\cdot n} = 1^n = 1$

check $x = \frac{1}{2}$: $\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n}\left(\frac{-1}{2}\right)^n = \sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent p-series (since $p = 1 \leq 1$) .

Interval of convergence : $\frac{1}{2} < x \leq \frac{3}{2}$

2.) (10 pts.) Use $a_n = \frac{f^{(n)}(a)}{n!}$ to find the first three (3) nonzero terms of the Taylor Series centered at $x = 4$ for $f(x) = \sqrt{x}$.

$$f(x) = \sqrt{x} \rightarrow f'(x) = \frac{1}{2}x^{-1/2} \rightarrow f''(x) = \frac{-1}{4}x^{-3/2} \quad ;$$

$$a_0 = f(4) = \sqrt{4} = 2 \,, \quad a_1 = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \,,$$

$$a_2 = \frac{f''(4)}{2!} = \frac{\frac{-1}{4}\cdot\frac{1}{4^{3/2}}}{2} = \frac{-1}{8}\cdot\frac{1}{8} = \frac{-1}{64} \;; \text{ then}$$

$$\sqrt{x} = a_0 + a_1(x-4) + a_2(x-4)^2 + \cdots$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \cdots$$

3.) (10 pts.) Use shortcuts to find the first three (3) nonzero terms of the Maclaurin Series for $f(x) = \cos\sqrt{x} \cdot \sin 2x$.

$$\cos\sqrt{x} \cdot \sin 2x = \left(1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \cdots\right)\left((2x) - \frac{(2x)^3}{3!} + \cdots\right)$$

$$= \left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \cdots\right)\left(2x - \frac{4}{3}x^3 + \cdots\right)$$

$$= 2x + \quad -\frac{4}{3}x^3 + \cdots$$

$$- \quad x^2 \quad + \quad \frac{2}{3}x^4 + \cdots$$

$$+ \frac{1}{12}x^3 \quad - \cdots$$

$$= 2x - x^2 - \frac{5}{4}x^3 + \cdots$$

4.) (10 pts.) Consider the function $f(x) = e^x$ on the closed interval $[0, 1]$. Estimate the value of $|R_4(x; 1)|$, the absolute value of the Lagrange form of the Taylor remainder (error). (You may assume that $e < 3$.)

$f(x) = e^x$, $f^{(n)}(x) = e^x$ for $n = 1, 2, 3, \cdots$, $0 \le x \le 1$, $n = 4$, and $a = 1$:

$$|R_4(x;1)| = \left| \frac{f^{(4+1)}(c_4)}{(4+1)!} \cdot (x-1)^{4+1} \right|$$

$$= \frac{|f^{(5)}(c_4)| \cdot |x-1|^5}{5!} = \frac{e^{c_4}}{5!} \cdot |x-1|^5$$

$$\le \frac{e^1 \cdot |0-1|^5}{5!} < \frac{3\,(1)}{5!} = \frac{1}{40}, \text{ so}$$

$$|R_4(x;1)| < \frac{1}{40} = 0.025$$

5.) (10 pts.) Let $\vec{u} = \overrightarrow{(1,-2)}$ and $\vec{v} = \overrightarrow{(3,4)}$. Find $proj_{\vec{v}}\vec{u}$.

$$proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \frac{3-8}{(\sqrt{9+16})^2}\vec{v}$$

$$= \frac{-5}{25}\vec{v} = \frac{-1}{5}\vec{v} = \overrightarrow{\left(\frac{-3}{5}, \frac{-4}{5}\right)}$$

6.) (10 pts.) Use a Taylor Polynomial to estimate the value of $\int_0^{1/2} \ln(1+x^2)\,dx$ with absolute error at most 1/1000.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \longrightarrow$$

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots \longrightarrow$$

$$\int_0^{1/2} \ln(1+x^2)\,dx = \int_0^{1/2}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots\right)dx \longrightarrow$$

$$= \left(\frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21} - \frac{x^9}{36} + \cdots\right)\Big|_0^{1/2}$$

$$= \frac{1}{2^3\cdot 3} - \frac{1}{2^5\cdot 10} + \frac{1}{2^7\cdot 21} - \frac{1}{2^9\cdot 36} + \cdots$$

(a conv. alt series with a_n +, ↓, and $\lim_{n\to\infty} a_n = 0$)

$$= \frac{1}{24} - \frac{1}{320} + \frac{1}{2688} - \cdots \text{ , so}$$

$$\frac{1}{2688} < \frac{1}{1000}$$

$$\int_0^{1/2} \ln(1+x^2)\,dx \approx \frac{1}{24} - \frac{1}{320} \approx 0.03854 \text{ with}$$

absolute error (alt. series) $< \frac{1}{2688} < \frac{1}{1000}$.

7.) Line L_1 : $\begin{cases} x = 1+t \\ y = 2t \\ z = -1+3t \end{cases}$ and line L_2 : $\begin{cases} x = 3+2s \\ y = 1+s \\ z = -2-s \end{cases}$ intersect.

a.) (5 pts.) Find the point (x, y, z) of intersection.

$$\left.\begin{aligned} 1+t &= 3+2s \\ 2t &= 1+s \end{aligned}\right\} \quad t = 2+2s \to 2(2+2s) = 1+s$$

$$\to 4+4s = 1+s \to 3+3s = 0 \to$$

$s = -1 \to$ pt. of $\cap$ is $(x,y,z) = (1,0,-1)$

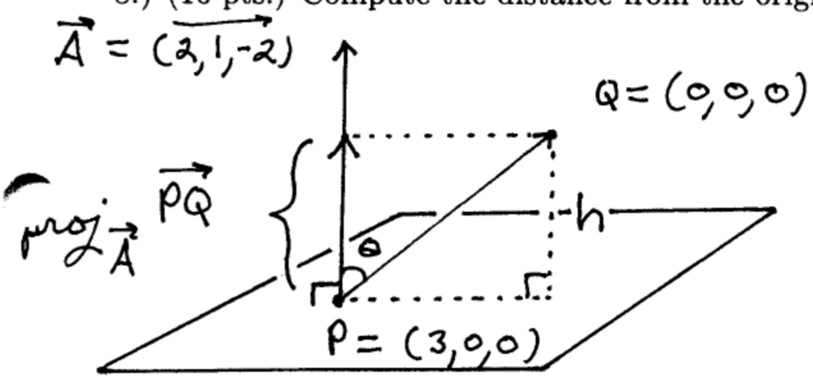
b.) (5 pts.) Find the angle θ between the lines.

$\vec{A} = \overrightarrow{(1,2,3)}$ is $\parallel$ to L_1; $\vec{B} = \overrightarrow{(2,1,-1)}$ is $\parallel$ to L_2; angle between $\vec{A}$ and $\vec{B}$ is given by

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{2+2-3}{\sqrt{14}\ \sqrt{6}} = \frac{1}{\sqrt{84}} \to$$

$$\theta = \arccos\left(\frac{1}{\sqrt{84}}\right) \approx 83.7^\circ$$

8.) (10 pts.) Compute the distance from the origin $(0,0,0)$ to the plane $2x + y - 2z = 6$.



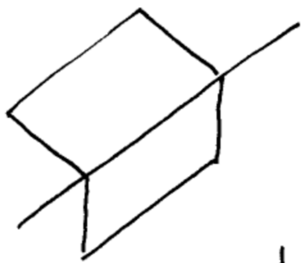
note that point $P = (3,0,0)$ is on plane and vector $\vec{A} = \overrightarrow{(2,1,-2)}$ is $\perp$ to plane. So distance from pt. $Q = (0,0,0)$ to plane is given by

$$h = |\text{proj}_{\vec{A}} \overrightarrow{PQ}|$$

$$= |\overrightarrow{PQ}|\cdot|\cos\theta| = \cancel{|\overrightarrow{PQ}|}\cdot\frac{|\overrightarrow{PQ}\cdot\vec{A}|}{\cancel{|\overrightarrow{PQ}|}|\vec{A}|}$$

$$= \frac{|\overrightarrow{(-3,0,0)}\cdot\overrightarrow{(2,1,-2)}|}{\sqrt{9}} = \frac{|-6+0+0|}{3} = 2$$

9.) (10 pts.) The two planes $x + 2z = 1$ and $x + y - z = 0$ intersect in a line L. Find a parametric representation for this line.



$\vec{A} = \overrightarrow{(1,0,2)}$ is $\perp$ to $x + 2z = 1$;

$\vec{B} = \overrightarrow{(1,1,-1)}$ is $\perp$ to $x + y - z = 0$; so

$\vec{A} \times \vec{B}$ is $\parallel$ line L :

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{vmatrix} = (0-2)\vec{i} - (-1-2)\vec{j} + (1-0)\vec{k}$$
$$= \underline{\underline{-2\vec{i} + 3\vec{j} + \vec{k}}}$$

; now find a point on the line :

$$\left.\begin{array}{l} x + 2z = 1 \\ x + y - z = 0 \end{array}\right\} \begin{array}{l} x + 2z = 1 \\ x + y = z \end{array} \longrightarrow \quad x + 2(x+y) = 1 \longrightarrow \quad 3x + 2y = 1 ;$$

let $x = 1$, $y = -1 \rightarrow z = 0$ so $\underline{\underline{(1,-1,0)}}$ is on L;

$$\text{line } L : \begin{cases} x = 1 - 2t \\ y = -1 + 3t \\ z = 0 + t \end{cases}$$

10.) Let $\vec{A} = \overrightarrow{(3, 0, -2)}$ and $\vec{B} = \overrightarrow{(0, -1, 1)}$.

a.) (5 pts.) Find the area of the parallelogram formed by placing these vectors tail-to-tail.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -2 \\ 0 & -1 & 1 \end{vmatrix} = (0-2)\vec{i} - (3-0)\vec{j} + (-3-0)\vec{k}$$
$$= -2\vec{i} - 3\vec{j} - 3\vec{k}$$

, so area of parallelogram is

$$|\vec{A} \times \vec{B}| = \sqrt{4 + 9 + 9} = \sqrt{22}$$

b.) (5 pts.) Find an equation of the plane containing the point $(1, -2, 3)$ and which is parallel to both vectors.

$\vec{A} \times \vec{B}$ is $\perp$ to both vectors, so

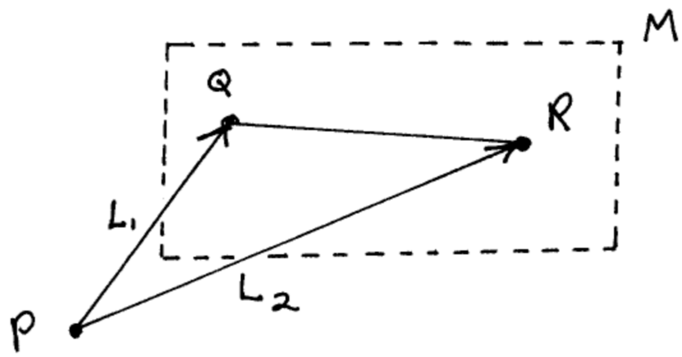
$\vec{A} \times \vec{B} = \overrightarrow{(-2,-3,-3)}$ is $\perp$ to the plane;

equation for plane is

$$-2(x-1) - 3(y-(-2)) - 3(z-3) = 0 \longrightarrow$$
$$-2x + 2 - 3y - 6 - 3z + 9 = 0 \longrightarrow$$
$$2x + 3y + 3z = 5$$

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Consider the lines $L_1 : \begin{cases} x = t \\ y = 2t \\ z = 1 - t \end{cases}$ and $L_2 : \begin{cases} x = 1 - s \\ y = 2 + s \\ z = 2s \end{cases}$ and the plane $M : 10x - 2y + 3z = 0$. Line L_1 intersects plane M at point Q. Line L_2 intersects plane M at point R. Lines L_1 and L_2 intersect at point P. Compute the area of triangle PQR.



Find pt. Q: $10(t) - 2(2t) + 3(1-t) = 0 \rightarrow$
$10t - 4t + 3 - 3t = 0 \rightarrow 3t + 3 = 0 \rightarrow t = -1 \rightarrow$
$\underline{Q = (-1, -2, 2)}$;

Find pt. R: $10(1-s) - 2(2+s) + 3(2s) = 0 \rightarrow$
$10 - 10s - 4 - 2s + 6s = 0 \rightarrow 6 - 6s = 0 \rightarrow s = 1 \rightarrow$
$\underline{R = (0, 3, 2)}$;

Find pt. P: $\left.\begin{array}{l} t = 1 - s \\ 2t = 2 + s \end{array}\right\} \rightarrow 2(1-s) = 2 + s \rightarrow$
$2 - 2s = 2 + s \rightarrow 0 = 3s \rightarrow$
$s = 0 \rightarrow \underline{P = (1, 2, 0)}$;

vectors $\vec{PQ} = \overrightarrow{(-2, -4, 2)}$ and $\vec{PR} = \overrightarrow{(-1, 1, 2)}$ form a triangle (half of parallelogram):

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -4 & 2 \\ -1 & 1 & 2 \end{vmatrix} = (-8-2)\vec{i} - (-4+2)\vec{j} + (-2-4)\vec{k}$$
$$= -10\vec{i} + 2\vec{j} - 6\vec{k}, \text{ ~~so~~}$$

$$\text{Area of } \Delta = \frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2}\sqrt{100 + 4 + 36} = \frac{1}{2}\sqrt{140}$$
$$= \frac{1}{2}\sqrt{4 \cdot 35} = \sqrt{35}$$