Math 21C
Kouba
SUMMARY:

Testing Infinite Series for Convergence / Divergence

1.) nth term test (for divergence only):
If $\lim_{n\to\infty} a_n \neq 0$, then the series
$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots \quad \text{diverges.}$$

2.) geometric series test: The series
$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots = \frac{1}{1-r} \quad \text{for } -1 < r < 1.$$
This series diverges for all other values of r.
Note also that $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1-r}$ for $r \neq 1$.

3.) p-series test: The series
$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$
a.) converges if $p > 1$.
b.) diverges if $p \leq 1$.

4.) integral test: Assume function f is cont., positive, and decreasing for $x \geq 1$, and consider the series $\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + \cdots$.

a.) If $\int_1^{\infty} f(x)\,dx$ converges, then the series converges.

b.) If $\int_1^{\infty} f(x)\,dx$ diverges, then the series diverges.

Let $R_n = f(n+1) + f(n+2) + f(n+3) + \cdots$. Then

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx .$$

5.) <u>sequence of partial sums</u>: Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots$ and let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ be a partial sum.

a.) If $\lim_{n \to \infty} S_n = L$, then $\sum_{n=1}^{\infty} a_n = L$.

b.) If $\lim_{n \to \infty} S_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

6.) <u>comparison test</u>: Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$.

a.) If $0 \le a_n \le c_n$ and $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

b.) If $0 \le d_n \le a_n$ and $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

7.) <u>limit comparison test</u>: Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ with $a_n \ge 0$.

a.) If $\sum_{n=1}^{\infty} c_n$ converges with $c_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{c_n} = L$ (a nonzero number), then $\sum_{n=1}^{\infty} a_n$ converges. If $L=0$, then $\sum_{n=1}^{\infty} a_n$ converges. If $L=+\infty$, then no conclusion can be made.

b.) If $\sum_{n=1}^{\infty} d_n$ diverges with $d_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{d_n} = L$ (a nonzero number), then $\sum_{n=1}^{\infty} a_n$ diverges. If $L = +\infty$, then $\sum_{n=1}^{\infty} a_n$ diverges. If $L=0$, then no conclusion can be made.

8.) <u>alternating series test</u>: Assume that a_n is positive and decreasing with $\lim_{n\to\infty} a_n = 0$. Then the series

$$\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n + \cdots$$

converges. Let

$$R_n = (-1)^{n+1} a_{n+1} + (-1)^{n+2} a_{n+2} + \cdots .$$

Then $|R_n| < a_{n+1}$.