1. Determine whether the following series converge. Specify the convergence tests you use.

(a) \[ \sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2 + \sqrt{n}} \]

ANS: \[ \sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2 + \sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n^2} \] which diverges by the p-series test so by the comparison test the original series diverges also.

(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n} \]

ANS: \[ \lim_{n \to \infty} \frac{\ln(n)}{n} = 0 \] by L'Hôpital's rule and so the series converges by the alternating series test.

2. For each of the following find an upper bound for the error resulting from estimating the infinite sum with just the first 5 terms.

(a) \[ \sum_{n=1}^{\infty} \frac{2n}{(1+x^2)^n} \]

ANS: The integral test error bound gives a maximum error of \[ \int_{5}^{\infty} \frac{2x}{(1+x^2)^n} \, dx = \frac{1}{1 + \frac{1}{x^2}} \] using the substitution \( u = 1 + x^2 \).

(b) \[ \sum_{n=1}^{\infty} (-1)^n \frac{3}{n^2} \]

ANS: The alternating series error bound gives a maximum error of \( \frac{3}{n^2} \). Alternatively the exact formula for geometric series gives an exact error of \( \frac{3}{27} \left( \frac{1}{1 - x} \right) \).

3. Determine the values of \( x \) for which the following series converges. Be sure to check the end points of the interval. \( \sum_{n=1}^{\infty} \sqrt{n \frac{1}{5^n}} \)

ANS: By the ratio test write \( p = \lim_{n \to \infty} \frac{x \sqrt{n}}{5 \sqrt{n+1}} = \frac{x}{5} \). If \(|p| < 1\) or \(-5 < x < 5\) the series converges (absolutely even) and if \(|p| > 1\) or \(|x| > 5\) it diverges. If \( x = \pm 5 \) then \( \lim_{n \to \infty} (\pm \sqrt{n \frac{1}{5^n}}) \) does not exist so the series diverges by n-th term test.

4. Find the first three nonzero terms of the Taylor series about \( x = 0 \) for the following function. \( f(x) = \cos(2x) - x \sin(x) \)

ANS: \[ f(x) = (1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4 + \ldots) - x(x - \frac{1}{6}x^3 + \ldots) = 1 + (2 - 1)x^2 + (\frac{16}{24} + \frac{1}{6})x^4 + \ldots \]

5. Find the ground speed (magnitude of the velocity vector) of a fly if the wind is blowing the fly 4 mi/hr northeast while the fly is flying 3 mi/hr west.

ANS: The velocity vector is \( \frac{1}{\sqrt{2}} \langle 1, 1 \rangle + \langle -3, 0 \rangle \) so the speed is \( \sqrt{(2\sqrt{2} - 3)^2 + 8} \).

6. Find an equation for the plane through the points \((1,1,1), (1,2,3)\) and \((-1,0,3)\).
10. Find all the local maxima, local minima and saddle points of the function $f(x, y) = 3\sqrt{4 - x^2 - y^2}$.

(a) Find and sketch the domain of $f$.
\textbf{ANS:} The disk of radius 2 centered at the origin.

(b) Find and sketch the range of $f$.
\textbf{ANS:} The interval $[0, 6]$.

(c) Describe the surface $z = f(x, y)$.
\textbf{ANS:} The top half of an ellipsoid.

7. Consider the function $f(x, y) = 3\sqrt{4 - x^2 - y^2}$.

(a) Find and sketch the domain of $f$.
\textbf{ANS:} The disk of radius 2 centered at the origin.

(b) Find and sketch the range of $f$.
\textbf{ANS:} The interval $[0, 6]$.

(c) Describe the surface $z = f(x, y)$.
\textbf{ANS:} The top half of an ellipsoid.

8. Consider again the surface $z = 3\sqrt{4 - x^2 - y^2}$. Find a parametric equation for the line normal to the surface at the point with $x = y = 1$.
\textbf{ANS:} The surface satisfies $g(x, y, z) = x^2 + y^2 + \frac{z^2}{4} - 4 = 0$ so the direction of the line is $\nabla g = (2x, 2y, \frac{z}{2})$ at the point $(1, 1, \sqrt{2})$. A parametric equation is therefore $(1 + 2t, 1 + 2t, 3\sqrt{2} + 2\sqrt{2}t)$.

9. Laplace’s equation for heat in a plate is satisfied by $f(x, y)$ if $f_{xx} + f_{yy} = 0$. Determine whether each of the following satisfy Laplace’s equation.

(a) $f(x, y) = e^{-2y} \cos(3x)$
\textbf{ANS:} $f_{xx} = -9e^{-2y} \cos(3x)$ and $f_{yy} = 4e^{-2y} \cos(3x)$ so Laplace’s equation is not satisfied.

(b) $f(x, y) = \ln(x^2 + y^2)$
\textbf{ANS:} $f_{xx} = \frac{d}{dx} \left( \frac{2x}{x^2 + y^2} \right) = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$ and similarly $f_{yy} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$ so Laplace’s equation is satisfied.

10. Find all the local maxima, local minima and saddle points of the function $f(x, y) = x^3 - y^3 - 2xy + 6$.
\textbf{ANS:} Every such point is critical so use the first derivative test and find points $(x, y)$ for which $0 = \nabla f = (3x^2 - 2y, 3y^2 - 2x)$. This requires that $x = \frac{2}{3}y^2$ and $y = \frac{3}{2}x^2$ so $y = \frac{3}{2}(-\frac{2}{3}y^2)^2 = \frac{27}{8}y^4$ and either $y = 0$ so $x = 0$ or $y = \frac{4}{3}$ so $x = -\frac{2}{3}$. Thus there are two critical points $(0, 0)$ and $(-\frac{2}{3}, \frac{4}{3})$. To determine their type use the second derivative test and compute $f_{xx} = 6x$, $f_{yy} = -6y$ and $f_{xy} = f_{yx} = -2$ so that $H = f_{xx}f_{yy} - f_{xy}f_{yx} = 36xy - 4$. At the point $(0, 0)$ one has $H = -4 < 0$ so the point is a saddle point. At the point $(-\frac{2}{3}, \frac{4}{3})$ one has $H = 12 > 0$ so the point is a local extremum and further $f_{xx} = -4 < 0$ so the point is a local maximum.
11. Find the maximum value of the function \( f(x, y, z) = x - 2y + 3z \) on the sphere \( x^2 + y^2 + z^2 = 14 \).

**ANS:** This is a constrained optimization so use the **Lagrange multiplier** method with \( f \) as given and \( g = x^2 + y^2 + z^2 - 14 \) so that at the maximum one has \( \nabla f = \langle 1, -2, 3 \rangle = \lambda \nabla g = \lambda \langle 2x, 2y, 2z \rangle \). This means that \( x = \frac{1}{\lambda} \lambda^{-1}, y = -\lambda^{-1} \) and \( z = \frac{3}{2} \lambda^{-1} \) so using \( g = 0 \) gives \( \frac{1}{4} + 1 + \frac{9}{4} \lambda^{-2} = \frac{14}{4} \lambda^{-2} = 14 \) so \( \lambda^{-1} = 2 \) and the maximum value is \( f(1, -2, 3) = 14 \).

12. (Optional extra credit problem.) Evaluate the sum \( \sum_{n=0}^{\infty} \frac{1}{(4n)!} \).

**ANS:** Consider the Maclaurin series for \( \frac{1}{4} e^x + \frac{1}{4} e^{-x} + \frac{1}{2} \cos(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!} \)
so the sum is \( \frac{e}{4} + \frac{1}{4e} + \frac{\cos(1)}{2} \).