Math 250A Homework 3, due 10/22/2021

1) Let $G$ be a finite group and $H$ a subgroup. Let $P_H$ be a $p$-Sylow subgroup of $H$. Prove that there is a $p$-Sylow subgroup $P$ of $G$ such that $P_H = P \cap H$.

2) Let $G$ be a group of order $p^3$ where $p$ is prime, and $G$ is not abelian. Let $Z$ be its center. Let $C$ be a cyclic group of order $p$.
   a) Show that $Z \cong C$ and $G/Z \cong C \times C$.
   b) Show that every subgroup of $G$ of order $p^2$ contains $Z$ and is normal.
   c) Suppose $x^p = 1$ for all $x$ in $G$. Show that $G$ contains a normal subgroup $H \cong C \times C$.

3) a) Prove that one of the Sylow subgroups of a group of order 40 is normal.
   b) Prove that one of the Sylow subgroups of a group of order 12 is normal.

4) Let $G$ be a finite group, and let $r$ be the number of conjugacy classes of $G$. Show that
   
   $$|\{(a, b) \in G \times G \mid ab = ba\}| = r|G|.$$ 

5) Let $G$ be a finite group, let $N \trianglelefteq G$, and let $P$ be a Sylow subgroup of $N$. Show that $G = N_G(P)N$, where $N_G(P)$ denotes the normalizer of $P$ in $G$.

6) a) Let $G$ be a group such that $p, q$ are two distinct prime divisors of $|G|$. Suppose that $P$ is the only $p$-Sylow subgroup of $G$ and $Q$ is the only $q$-Sylow subgroup of $G$. Show that the elements of $P$ commute with the elements of $Q$.
   b) Show that any group of order 45 is abelian.

7) Show that every Sylow subgroup is normal in $G$ if and only if $G$ is the direct product of its Sylow subgroups.

8) Let $H$ be a cyclic group and let $N$ be an arbitrary group. If $\phi$ and $\psi$ are injective homomorphisms from $H$ to $\text{Aut}(N)$ such that $\phi(H) = \psi(H)$, show that $N \rtimes_\phi H \cong N \rtimes_\psi H$. 

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