Math 250A Homework 8, due 10/26/2012

1) Let $K/F$ be a Galois extension with $[K : F] = n$. Show that if $p|n$ is a prime then there is a subfield $L$ of $K$ with $[K : L] = p$.

2) Let $K/F$ be a Galois extension with $\text{Gal}(K/F) \cong A_4$. Show that there is no intermediate field $M$ of the extension $K/F$ such that $[M : F] = 2$.

3) Show that if $K/F$ is a Galois extension such that there are no proper intermediate fields between $K$ and $F$, then $[K : F]$ is a prime number. Is this still true if $K/F$ is not a Galois extension?

4) Let $K/F$ be a Galois extension and $\alpha \in K$ and $H = \text{Gal}(K/F(\alpha))$. Let $[K : F] = n$ and $[F(\alpha) : F] = r$. Suppose that $\{\tau_1, \cdots, \tau_r\}$ is a set of left coset representatives of $H$ in $\text{Gal}(K/F)$. Show that the minimal polynomial of $\alpha$ over $F$ is given by

$$m(x) = \prod_{i=1}^{r}(x - \tau_i(\alpha))$$

and show that

$$\prod_{\sigma \in \text{Gal}(K/F)} (x - \sigma(\alpha)) = m(x)^{n/r}$$

5) Let $K/F$ be a Galois extension with $\text{Gal}(K/F) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\text{char}(F) \neq 2$. Show that $K = F(\sqrt{\alpha}, \sqrt{\beta})$ for some $\alpha, \beta \in F$.

6) Let $K$ be the splitting field of $x^8 - 1$ over $\mathbb{Q}$. Find $\text{Gal}(K/\mathbb{Q})$ and describe all intermediate fields of $K/\mathbb{Q}$.

7) Let $S = \{\sqrt{p} \mid p \text{ a prime}\}$ and $K = \mathbb{Q}(S)$. For $\sigma \in \text{Aut}(K/\mathbb{Q})$ define

$$Y_\sigma = \{\sqrt{p} \mid \sigma(\sqrt{p}) = -\sqrt{p}\}$$

Show the following:

(i) $K$ is normal and separable over $\mathbb{Q}$

(ii) If $Y_\sigma = Y_\tau$, then $\sigma = \tau$

(iii) If $Y$ is a subset of $S$ then there exists $\sigma \in \text{Aut}(K/\mathbb{Q})$ such that $Y = Y_\sigma$
