1. (9 points each) Determine whether each of the following series converges or diverges. Write clear and complete solutions. This should include the name of any series convergence test that you use.

(a) \[ \sum_{n=1}^{\infty} \sqrt{\frac{n^3}{n^3+1}} \]

**ANS:** \( \lim_{n \to \infty} \left( \sqrt{\frac{n^3}{n^3+1}} \right) = \lim_{n \to \infty} \left( \frac{1}{\sqrt{1+n^{-3}}} \right) = 1 \) so by the \( n \)-th term test the series diverges.

(b) \[ \sum_{n=1}^{\infty} \frac{25}{100^n} \]

**ANS:** This series is geometric with \( r = \frac{1}{100} \) so by the geometric series test it converges.

(c) \[ \sum_{n=1}^{\infty} \frac{1}{(n+5)\sqrt{n}} \]

**ANS:** Take \( b_n = \frac{1}{n^{3/2}} \) which is a convergent series by the \( p \)-series test and notice that \( 0 \leq a_n \leq b_n \). Thus the series converges by the comparison test.

(d) \[ \sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n+1}) \]

**ANS:** This series is telescoping and the partial sums are \( s_N = 1 - \sqrt{N+1} \) which form a divergent sequence. Thus the series diverges by the partial sums or telescoping test.

(e) Use the integral test.

\[ \sum_{n=1}^{\infty} \frac{1}{(n+1)\ln^2(n+1)} \]
ANS: Here $a_n = f(n)$ with $f(x) = \frac{1}{(x+1)\ln^3(x+1)}$ a positive decreasing function. This function can be integrated by substitution with $u = \ln(x + 1)$ so that $du = \frac{dx}{x+1}$ and $\int_1^\infty f(x)dx = \int_{\ln(2)}^{\infty} u^{-3}du = \frac{1}{2\ln^2(2)}$. Since the integral converges, so does the series.

(f) Use the limit comparison test.

$$\sum_{n=1}^{\infty} \frac{n + \sin(n)}{\sqrt{n^3 + 1}}$$

ANS: Choose $b_n = \frac{1}{n^\alpha}$ which diverges by the $p$-series test. Compute

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n + \sin(n)}{\sqrt{1 + \frac{1}{n^3}}} = 1.$$ Since this is a nonzero constant the given series also diverges.

(g)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n + 1}$$

ANS: Use the alternating series test with $b_n = \frac{1}{2n+1}$ which is positive and decreasing and has $\lim_{n \to \infty} (b_n) = 0$ so that the given alternating series converges.

(h)

$$\sum_{n=1}^{\infty} \ln^{-n}(n + 1)$$

ANS: Use the root test and compute that $\lim_{n \to \infty} a_n^{\frac{1}{n}} = \lim_{n \to \infty} \frac{1}{\ln(n+1)} = 0$ which is less than 1 so the given series converges.

2. (8 points) Write the repeating decimal $18.\overline{18} = 18.18181818\ldots$ as a fraction.

ANS: $\sum_{n=1}^{\infty} 18(\frac{1}{100})^{n-1} = \frac{18}{1 - \frac{1}{100}}$

3. (10 points each) Consider the following convergent infinite series. How many terms should be used in a partial sum to estimate the infinite sum with an absolute error of no more than $\frac{1}{18}$?

(a)

$$\sum_{n=1}^{\infty} \frac{1}{3^n-1}$$
ANS: Since the series is a geometric series with $a = 1$ and $r = \frac{1}{3}$, one has that $|R_n| = \frac{n}{1-r} = \frac{1}{2.3^{n-1}}$ and to get $\frac{1}{2.3^{n-1}} \leq \frac{1}{18}$ it suffices to take $n = 3$.

(b) \[ \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \]

ANS: Since $a_n = f(n)$ with $f(x) = x^{-\frac{3}{2}}$ a positive decreasing function one has that $|R_n| \leq \int_{n}^{\infty} f(x)dx = \frac{2}{\sqrt{n}}$ and to get $\frac{2}{\sqrt{n}} \leq \frac{1}{18}$ it suffices to take $n = 36^2$.

4. (10 points. Optional extra credit problem.) A bouncy ball is bounced down an infinite flight of stairs with each step one foot high. The ball bounces once on each step. The number of feet $a_{n+1}$ that it falls to the $(n+1)$-st step is half of the number of feet $a_n$ that it falls to the $n$-th step (because of the bounce) plus 1 (because it falls to the next step) and $a_1 = 1$ (because it is rolled off the top).

Find the limit of the sequence $\{a_n\}$ (the eventual distance the ball falls to the next step) or show that it diverges.

ANS: $a_{n+1} = 1 + \frac{1}{2}a_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$ which is a geometric series so $\lim_{n \to \infty} a_n = 2$. 