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Math 201A 2018 Take-Home Midterm
Due 10:30 November 20.

Please do not consult with others. Please return this exam either physically (to my box or office or in class) or email it to me.

(1) Consider a sequence \(\{(X_i, d_i)\}_{i=0}^{\infty}\) of compact, diameter-1 metric spaces (that is \(\sup_{x,y \in X_i} d_i(x, y) = 1\) for every \(i\)) and define \(X = \prod_i X_i\) the product as a set and \(d(\{x_i\}, \{y_i\}) = \sup_i d_i(x_i, y_i)\).

Show that \((X, d)\) is

(a) a metric space,
(b) complete,
(c) not separable,
(d) not homeomorphic to the product topological space.

(2) Find functions \(K \in C((-1, 1)^2)\) and \(f \in C(-1, 1)\) so that \((T_Kf)(x) = \int_{-1}^{1} K(x, y)f(y)dy\) is finite for every \(x \in (-1, 1)\) but \(T_Kf\) is not continuous at 0.

(3) Show that there is a unique Borel measure \(\mu\) on \([0, 1]\) with the usual metric topology so that for every non-negative integer \(n\) one has \(\int d\mu(x)x^n = 3 + \frac{1}{n^2}\) and compute \(\mu([0, 1/2])\).

(You may assume that positive linear functions from \(C[0, 1]\) to \(\mathbb{R}\) are \(\| \cdot \|_{[0,1]}\)-continuous).

(4) For any \(p, q \in (1, \infty)\) consider the vector space \(X^{p,q} = L^p \mathbb{R} \cap L^q \mathbb{R}\) and the product Banach space \(Y^{p,q} = L^p \mathbb{R} \times L^q \mathbb{R}\) with norm \(\| \cdot \|_Y\) along with the diagonal map \(\Delta : X^{p,q} \to Y^{p,q}\) with \(\Delta(f) = (f, f)\). \(X^{p,q}\) has three norms: \(\| \cdot \|_p, \| \cdot \|_q\) and \(\| \cdot \|_{Y^*}\) where \(\|f\|_{Y^*} = \|\Delta(f)\|_Y\).

For which (if any) of these norms is \(X\) complete?

(5) For three of the following show that there is a unique solution \(u \in C[0, 1]\) and find an upper bound for \(u(1)\).

(a) \(u(x) = \frac{1}{4} \int_{0}^{1} e^{xz} u(z)dz + 4 \sin(2\pi x)\)

(b) \(u(x) = \frac{1}{4} \int_{0}^{x} e^{xz} u(z)dz + 4 \sin(2\pi x)\)

(c) \(u(x) = 4 \int_{0}^{1} e^{xz} u(z)dz + \frac{1}{4} \sin(2\pi x)\)

(d) \(u(x) = 4 \int_{0}^{x} e^{xz} u(z)dz + \frac{1}{4} \sin(2\pi x)\)

(6) Consider the absolute value function \(f \in L^2[-1, 1]\) with \(f(x) = |x|\) and the three dimensional subspace \(D = \{[a + bx + cx^2 + dx^3] \in L^2[-1, 1]\}\) of low degree polynomials.

(a) Find the \(\| \cdot \|_{L^2}\)-closest polynomial in \(D\) to \(f\).

(b) Is the same polynomial also a \(\| \cdot \|_{[-1,1]}\)-closest polynomial in \(D\) to \(f\)?