MAT 108 Homework 4 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 1.4 #5c, 7d, 9d, 11bde

5. (c) Let x and y be integers and suppose that x and y are both even. By definition, we have $x = 2k$ and $y = 2l$ for $k, l \in \mathbb{Z}$. Then the product xy is

$$xy = (2k)(2l) = (4kl)$$

where we've freely applied the associativity and commutativity of integer multiplication in our computation above. Since k and l are integers, their product kl is also an integer because $\mathbb{Z}$ is closed under multiplication. Therefore, $xy = 4(kl)$ is an integer multiple of 4, and hence xy is divisible by 4.

7. (d) Let a be an integer. We consider two cases:

- First, suppose a is odd. Then $a = 2k+1$ for some $k \in \mathbb{Z}$ and $a+1 = 2k+2$. Their product is

$$a(a+1) = (2k+1)(2k+2) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$$

where we freely apply the associativity and commutativity of integer addition and multiplication. Since k is an integer, $2k^2 + 3k + 1$ is also an integer and therefore $a(a+1)$ is an integer multiple of 2. Thus, $a(a+1)$ is even when a is odd.

- Now, suppose a is even. Then $a = 2k$ for some $k \in \mathbb{Z}$ and $a + 1 = 2k + 1$. Their product is

$$a(a+1) = (2k)(2k+1) = 4k^2 + 2k = 2(2k^2 + k)$$

where we again freely apply the associativity and commutativity of integer addition and multiplication. Since k is an integer, $2k^2 + k$ is also an integer and therefore $a(a+1)$ is an integer multiple of 2. Thus, $a(a+1)$ is even when a is even.

Since a is either even or odd, we have shown that $a(a+1)$ is even for any $a \in \mathbb{Z}$.

9. (d) Let $x \in \mathbb{R}$ be a real number. Starting with the desired conclusion $2x + 5 < 11$, we have

$$\begin{aligned} 2x + 5 < 11 &\iff 2x - 6 < \\ &\iff x - 3 < 0 \end{aligned}$$

Therefore, if we assume $x^3 + 2x^2 < 0$, then we have

$$\begin{aligned} x^3 + 2x^2 < 0 &\iff x^2(x+2) < 0 \\ &\iff x^2 < 0 \text{ or } (x+2) < 0 \end{aligned}$$

Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, we must have $x + 2$¡0. Hence, $x - 3 < 0$ and therefore our above computation implies $2x + 5 < 11$.

11. (b) C. The claim is correct and the proof is largely correct. However, we cannot assume that b and c are the same number aq, as has been done in the attempted proof. The solution should use different variables when writing out divisibility conditions, e.g., $b = aq$ and $c = ar$.

(d) F. The attempted proof assumes the truth of the conclusion (i.e. the thing they're trying to show is true) at the very beginning. Really this is a proof of the converse statement "m odd implies m^2 odd".

(e) C. The claim is correct and the attempted proof has the right idea. However, there are many missing gaps. For example, the claim that $a^2(a+1)$ is an even number times an odd number is true, but needs much more careful consideration to be fully convincing.