MAT 108 Homework 3 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 1.5 #3f, 6e, 9, 12ace

3. Let $x, y \in \mathbb{Z}$. Prove by contraposition.

(f) If xy is odd, then x and y are both odd.

Solution: Let x and y be integers. Suppose that x and y are not both odd. Then at least one of x and y is even. Without loss of generality,[1] we will assume that x is even. By definition, $x = 2m$ for some integer $m \in \mathbb{Z}$. Then the product $xy = 2my$. Since y is also an integer, the product my is an integer as well. Therefore, xy is a multiple of 2, hence even. We have shown that if x and y are not both odd, then their product xy is even. Thus, if xy is odd, then x and y must both be odd.

6. Let $x, y \in \mathbb{Z}_{>0}$. Prove by contradiction.

(e) If $a < b$ and $ab < 3$, then $a = 1$.

Solution: Let a and b be positive integers satisfying $a < b$ and $ab < 3$, and assume towards a contradiction that $a \neq 1$. Since a is positive, we know by definition that $a > 0$. Therefore, $a \neq 1$ implies that $a > 1$.[2] Moreover, since a is an integer, $a > 1$ implies $a \geq 2$. Similarly, $a < b$ implies that $b \geq 3$. Therefore, $ab \geq 6$ by the properties of inequalities. But $ab \geq 6$ contradicts $ab < 3$, so our original assumption that $a \neq 1$ was false. Thus $a < b$ and $ab < 3$ implies $a = 1$.

9. Prove by contradiction that $n \in \mathbb{N}$ implies $\frac{n}{n+1} > \frac{n}{n+2}$.

Solution: Let $n \in \mathbb{N}$ and assume towards a contradiction that $\frac{n}{n+1} \leq \frac{n}{n+2}$. Cross multiplying, we have that $(n+2)n \leq (n+1)n$. If we apply the distributive property, our inequality becomes $n^2+2n \leq n^2+n$. We can then subtract n^2 from both sides to get $2n \leq n$. Since n is a natural number, dividing both sides by n implies $2 \leq 1$. This is a contradiction. Therefore, our original assumption was false and

$$\frac{n}{n+1} > \frac{n}{n+2}.$$

12. Grade the given proofs

Grades:

(a) F. The statement "m^2 is not odd implies m is not odd" is not equivalent to the original claim.

(c) A. This proof is good so long as the proof writer has already proven elsewhere that the sum or difference of even numbers is even.

(e) C. The argument is correct, but the proof is missing a conclusion and the fact that $2j^2 + 2j + 2k^2$ is an integer has not been stated.

[1]The phrase 'without loss of generality' means that we could have chosen y to be even and made the exact same argument. The other thing that I haven't said explicitly is that this argument also works if both x and y are even. Of course, it's completely acceptable to argue case by case (First, if x is even..., Now if y is even...).

[2]You might want to cite the trichotomy property here if you were particularly careful.