MAT 108 Homework 3 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 1.5 #10

Section 1.6 #4i

Section 1.7 #5e, 11adg

10. Prove that $\sqrt{5}$ is not a rational number.

Solution:

Assume towards a contradiction that $\sqrt{5}$ is a rational number. Then, by defintion, we have $\sqrt{5} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. Suppose that $\frac{p}{q}$ is in reduced form, so that $q \not| p$. If $\frac{p}{q}$ were not in reduced form, we could always divide by the greatest common denominator to write $\sqrt{5}$ in reduced form. Written as a rational number, $\sqrt{5} = \frac{p}{q}$ implies $\sqrt{5}q = p$, or $5q^2 = p^2$. We know from lecture that for any integer a, if a^2 is odd, then a is odd, and if a^2 is even, then a is even. If we consider the case where q^2 is odd, then $5q^2 = p^2$ is also odd because the product of odd numbers is odd. By the fact we just recalled, this implies that both q and p are odd. If we instead consider the case where q^2 is even, then $5q^2 = p^2$ is also even, since the product of two even numbers is again even. In this case, we must have that both q and p are even. Therefore, we must have either p and q are both odd or they are both even. This contradicts our fact from lecture that a fraction in reduced form must have odd numerator and even denominator or even numerator and odd denominator. Thus, $\sqrt{5}$ is not a rational number, i.e., $\sqrt{5} \in \mathbb{R} \backslash \mathbb{Q}$.

4. Find a proof or counterexample

(i) For every positive real number x there is a positive real number y with the property that if $y < x$, then for all positive real numbers z, $yz \geq z$.

Solution: The number $x = 1$ is a counterexample. Every positive real number $y < 1$, when multiplied by a positive real number z yields $yz < z$.

5. Prove that

(e) for every rational number z and every irrational number x, there exists a unique irrational number y such that $x + y = z$.

Solution: We first start with a short lemma. Claim: The sum of a rational and an irrational number is irrational.

Proof: Let $x \in \mathbb{R} \backslash \mathbb{Q}$ be irrational and $z \in \mathbb{Q}$ be rational. Assume by way of contradiction that $x + z \in \mathbb{Q}$. From quiz 3, we know that the sum of any two rational numbers is again rational, and the same argument can be applied to show that the difference is also rational. Therefore, $x = (x + z) - z$ is the difference of two rational numbers, hence rational. This contradicts the fact that x is irrational, so therefore the sum of a rational and an irrational is irrational.

Now, we prove the initial claim:

Let $z \in \mathbb{Q}$ be a rational number and $x \in \mathbb{R} \backslash \mathbb{Q}$ be an irrational number. By the properties of real numbers, there exists a unique additive inverse $-x$ of x such that $x + -x = 0$. Our claim above then implies that $-x$ must be irrational because we know that 0 is not irrational. The sum $-x + z$ must also be rational, again by our claim above. Moreover, we have

$$x + (-x + z) = (x - x) + z = 0 + z = z$$

where we've freely made use of the associativity of addition of real numbers. Thus, we have produced an irrational number $y := -x + z$ such that $x + y = z$. The fact that y is unique follows from uniqueness of additive inverses, as adding the unqiue additive inverse $-x$ to both sides of the equation $x + y = z$ uniquely determines $y = z - x$.

11. 'Grade' the following proofs:

(a) (see textbook for proof)

Solution: F. The attempted proof gives an example and justification for why that specific example satisfies the criteria. However, it does not show uniqueness. In fact, the claim is false and the example given is not unique. For example, the number 251 is also a perfectly good example.

(d) (see textbook for proof)

Solution: A. Solid proof by contradiction. A somewhat harsh grader might say C due to the fact that the proof does not specify nonzero denominators or that $2b$ is an integer.

(g) (see textbook for proof)

Solution: F. This is a reasonable proof of the statement "n prime and $n+5$ prime implies $n=2$, but the claim is false as written and the attempted proof does not examine the $n+12$ prime case.