MAT 108 Homework 8 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 1.8 #7ac, 17e, 21cdf

7. Let $a, b, c \in \mathbb{N}$ and $\gcd(a, b) = d$. Prove that

(a) a divides b if and only if $d = a$.

Solution: Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = d$ and suppose that $a|b$. Then we we have $b = ka$ for some $k \in \mathbb{N}$. Since d is the gcd of a and b, we can also write $a = md$ and $b = nd$ where m and n are natural numbers. Putting these equalities together, we have

$$b = ka = k(md).$$

Since k is an natural number, the above equalities imply that $md|b$ and since $a = md$, we certainly have $md|a$. By the definition of gcd, $md \leq d$, so we must have $m \leq 1$ and $m \in \mathbb{N}$, i.e., $m = 1$. Therefore, $a|b$ implies $d = a$.

Now, suppose instead that $d = a$. Then, by definition of gcd, $d|b$, so it immediately follows that $a|b$. Thus, $a|b$ if and only if $d = a$.

(c) if c divides a and c divides b, then $\gcd(a/c, b/c) = \frac{d}{c}$. In particular, $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$.

Solution:

Let $a, b, c \in \mathbb{N}$ with $\gcd(a, b) = d$ and suppose that c divides both a and b. Since c is a common divisor of both a and b, and $\gcd(a, b) = d$, we know that $c \leq d$. Our first goal is to show that $c|d$: By definition of the gcd, we have $m_1 d = a$ and $m_2 d = b$ for $m_1, m_2 \in \mathbb{N}$. Therefore, we can write $\frac{m_1 d}{c} = \frac{a}{c}$ and $\frac{m_2 d}{c} = \frac{b}{c}$, where $\frac{a}{c}$ and $\frac{b}{c}$ are natural numbers. If we suppose that $c \not| d$, then $\frac{m_1}{c}$ and $\frac{m_2}{c}$ then must be natural numbers as well, since $\frac{m_i d}{c}$ is equal to a natural number and $c \not| d$. If we clear denominators on the right hand side of the equations, we have $\frac{m_1}{c}(cd) = a$ and $\frac{m_2}{c}(cd) = a$. Since $\frac{m_i}{c}$ is a natural number, it follows that cd divides both a and b. Since c does not divide d, we know that $c \neq 1$, so $cd > d$. But this contradicts the definition of gcd, so therefore we must have $c|d$.

Since $\frac{d}{c}$ is a natural number, the equations $\frac{m_1 d}{c} = \frac{a}{c}$ and $\frac{m_2 d}{c} = \frac{b}{c}$ imply that $\frac{d}{c}$ divides both $\frac{a}{c}$ and $\frac{b}{c}$. Therefore it only remains to show that any other factor of $\frac{a}{c}$ and $\frac{b}{c}$ is less than or equal to $\frac{d}{c}$ in order to prove the original statement.

Let $k \in \mathbb{N}$ be a common factor of both $\frac{a}{c}$ and $\frac{b}{c}$. Then we can write $n_1 k = \frac{a}{c}$ and $n_2 k = \frac{b}{c}$ for $n_1, n_2 \in \mathbb{N}$. Multiplying by c on both sides of these equations, we have $n_1(ck) = a$ and $n_2(ck) = b$, which tells us that $ck|a$ and $ck|b$. Therefore, by definition of gcd, $ck \leq d$. Dividing by c, we get $k \leq \frac{d}{c}$. Thus, we have shown that $\frac{d}{c}$ is a common factor of $\frac{a}{c}$ and $\frac{b}{c}$ and any other common factor is less than or equal to $\frac{d}{c}$. Hence, $\gcd(a/c, b/c) = \frac{d}{c}$.

If we set $c = d$, then the original statement immediately implies $\gcd(\frac{a}{d}, \frac{b}{d}) = d/d = 1$.

17. Let $a, b, c \in \mathbb{N}$ with $\gcd(a, b) = d$ and $\text{lcm}(a, b) = m$. Prove that

(e) for every natural number n, $\text{lcm}(an, bn) = mn$.

Solution: Let $a, b, n \in \mathbb{N}$ with $\text{lcm}(a, b) = m$. Then there exist $s, t \in \mathbb{N}$ such that $sa = tb = m$. Therefore, $(sa)n = (tb)n = mn$, so mn is a common multiple of an and bn. In order to show that mn is the *least* common multiple, consider some other common multiple k of an and bn. Then there exist natural numbers $s', t' \in \mathbb{N}$ such that $s'an = t'bn = k$. Since $s't$, and $t'b$ are integers, this expression tells us that $n|k$. Therefore, $s'a = t'b = \frac{k}{n}$. By definition, this implies that $\frac{k}{n} \geq m$, from which it follows that $k \geq mn$. Thus, mn is the lcm of an and bn, as desired.

21. 'Grade' the following proofs:

(c) (see textbook for proof)
Solution: C. The proof has all of the right ideas, but does not explicitly show divisibility by 3 for the two cases.

(d) (see textbook for proof)
Solution: A. Good proof by contradiction.

(f) (see textbook for proof)
Solution: A. Proof is good so long as we assume the result from 17f.