MAT 108 Homework 12 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 2.3 #4ab, 7b, 10b, 13, 19bcd

4. Find the union and intersection of these families, and prove your answers.

(a) $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$, where $A_n = \{4n, 4n+1, \dots, 5n\}$ for each natural number n.

Solution:

$$\bigcup_{n\in\mathbb{N}} A_n = \{4, 5, 8, 9, 10, 12, 13, 14, 15\} \cup \{n \in \mathbb{N} : n \geq 16\}.$$

$$\bigcap_{n\in\mathbb{N}} A_n = \emptyset.$$

Ask on Piazza or in James's office hours for proof(s)

(b) $\mathcal{A} = \{B_n : n \in \mathbb{N}\}$, where $B_n = \mathbb{N} - \{n, n+1\}$ for each natural number n.

Solution:

$$\bigcup_{n\in\mathbb{N}} B_n = \mathbb{N}$$

$$\bigcap_{n\in\mathbb{N}} B_n = \emptyset$$

Ask on Piazza or in James's office hours for proof(s)

7. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets, and let B be a set. Prove that

(b) $B \cup \bigcap_{\alpha\in\Delta} A_\alpha = \bigcap_{\alpha\in\Delta}(B \cup A_\alpha)$.

Solution:
Let $\mathcal{A}$ be as given in the statement of the problem and let $x \in B \cup \bigcap_{\alpha\in\Delta} A_\alpha$. Then $x \in A_\alpha$ for all $\alpha \in \Delta$ or $x \in B$. Therefore, $x \in B \cup A_\alpha$ for all $\alpha \in \Delta$. Thus, $x \in \bigcap_{\alpha\in\Delta}(B \cup A_\alpha)$.
Now, let $x \in \bigcap_{\alpha\in\Delta}(B \cup A_\alpha)$. Then, $x \in B \cup A_\alpha$ for all $\alpha \in \Delta$, which means $x \in B$ or $x \in A_\alpha$ for all α. Hence, $x \in B \cup \bigcap_{\alpha\in\Delta} A_\alpha$. Thus, since we have shown that the two sets contain each other, we have $B \cup \bigcap_{\alpha\in\Delta} A_\alpha = \bigcap_{\alpha\in\Delta}(B \cup A_\alpha)$, as desired

10. If $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ is a family of sets and $\Gamma \subseteq \Delta$, prove that

(b) $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcap_{\alpha\in\Gamma} A_\alpha$

Solution: Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets and let $\Gamma \subseteq \Delta$ be a subset of the indexing set Δ. Consider $x \in \bigcap_{\alpha\in\Delta} A_\alpha$. By definition, $x \in A_\alpha$ for all $\alpha \in \Delta$. Since $\Gamma \subseteq \Delta$, $x \in A_\alpha$ for all $\alpha \in \Gamma$ as well. Therefore, $x \in \bigcap_{\alpha\in\Gamma} A_\alpha$. Thus, we have shown $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcap_{\alpha\in\Gamma} A_\alpha$.

13. Give an example of an indexed collection of sets $\{A_n : n \in \mathbb{N}\}$ such that (i) each $A_n \subseteq (0, 1)$ and (ii) for all $m, n \in \mathbb{N}, A_m \cap A_n \neq \emptyset$, but $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.

Solution: Define the collection $\mathcal{A}$ of sets $A_n = \{(0, \frac{1}{n+1})\}$ for every $n \in \mathbb{N}$. A_n satisfies the following properties:

- Since $0 < \frac{1}{n+1} < 1$ for all $n \in \mathbb{N}$, each $A_n \subseteq (0, 1)$.
- For any $m, n \in \mathbb{N}$, $A_m \cap A_n = (0, \frac{1}{m}) \neq \emptyset$ for $m < n$.
- $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ because, for any $x \in (0, 1)$ there exists a n large enough that $\frac{1}{n} < x$, which implies $x \notin A_n$.[1]

19. 'Grade' the following proofs:

(b) (see textbook for proof)
Solution: C. The proof skips the step of stating that x is in some A_α because it is contained in the union. It's possible to make the case that that is a minor detail, but it is a key step in the proof.

(c) (see textbook for proof)
Solution: F. Proof is an example.

(d) (see textbook for proof)
Solution: A. It's not necessary to prove this by contradiction, but this proof is still solid.

[1] This follows the Archimedean property of real numbers I may have mentioned in section.