MAT 108 Homework 14 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 2.5 #1a, 3, 7b, 12, 14cde

1. Use the PCI to prove that

(a) every natural number greater than or equal to 11 can be written in the form $2s + 5t$ for some natural numbers s and t.

Solution: (base case) Let $n = 11$. Then we can write $11 = 2(3) + 5(1)$ where $3, 1 \in \mathbb{N}$. Similarly, we can write $12 = 2(1) + 5(2)$ for $1, 2 \in \mathbb{N}$. Therefore, the statement is true for $n = 11$ and $n = 12$.

(induction step) Assume that we can write any number $11 \leq k < n$ in the form $k = 2s + 5t$ for some $s, t \in \mathbb{N}$. We must show that we can write $n = 2s_0 + 5t_0$ for some $s_0, t_0 \in \mathbb{N}$. By the inductive hypothesis, we know that we can write $n - 2 = 2s_1 + 5t_1$ for some $s_1, t_1 \in \mathbb{N}$. Rearranging this expression, we have

$$n = 2s_1 + 5t_1 + 2 = 2(s_1 + 1) + 5t_1.$$

Since $s_1 + 1 \in \mathbb{N}$, we have shown that if the statement is true for $11 \leq k < n$, then the statement is true for $k = n$. Thus, by the PCI, the statement is true for all natural numbers $n \geq 11$.

3. Let $a_1 = 2, a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$. Prove that $a_n = 2^n$ for all natural numbers n.

Solution: (base case) Let $a_1 = 2, a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$. We have $a_1 = 2^1 = 2$ and $a_2 = 2^2 = 4$, so the statement is true for $n = 1$ and $n = 2$.

(induction step) Assume that $a_k = 2^k$ for all natural numbers $1 \leq k < n$. By definition, we can write $a_n = 5a_{n-1} - 6a_{n-2}$. Applying the inductive hypothesis, we have $a_{n-1} = 2^{n-1}$ and $a_{n-2} = 2^{n-2}$, so

$$a_n = 5(2^{n-1}) - 6(2^{n-2}) = 10(2^{n-2}) - 6(2^{n-2}) = 4(2^{n-2}) = 2^n.$$

Therefore, if the statement is true for all $k < n$, then the statement is true for $k = n$. Thus, by the PCI, $a_n = 2^n$.

7. Use the PCI to prove the following properties of Fibonacci numbers:

(b) $f_{n+6} = 4f_{n+3} + f_n$ for all natural numbers n.

Solution: (base case) Let f_n denote the nth Fibonacci number. Since $f_7 = 13, f_4 = 3$, and $f_1 = 1$, we have $f_7 = 4(3) + 1 = 13$, so the claim is true for $n = 1$. Similarly, $f_8 = 21, f_5 = 5$, and $f_2 = 1$, and $f_8 = 4(5) + 1 = 21$, so the claim is true for $n = 2$ as well.

(induction step) Assume that $f_{k+6} = 4f_{k+3} + f_k$ for all $1 \leq k < n$. We must show that $f_{n+6} = 4f_{n+3} + f_n$. By the definition of Fibonacci numbers, we can write f_{n+6} as $f_{n+5} + f_{n+4}$. Applying the inductive hypothesis to each of these, we have

$$f_{n+6} = f_{n+5} + f_{n+4} = (4f_{n+2} + f_{n-1}) + (4f_{n+1} + f_{n-2}) = 4(f_{n+2} + f_{n+1}) + (f_{n-1} + f_{n-2}) = 4f_{n+3} + f_n.$$

Therefore, if the statement is true for all $k < n$, the statement is true for $k = n$ as well. Thus, by the PCI, $f_{n+6} = 4f_{n+3} + f_n$.

12. Let the "Fibonacci-2" numbers g_n be defined as follows:

$$g_1 = 2, g_2 = 2, \text{ and } g_{n+2} = g_{n+1}g_n \text{ for all } n \geq 1.$$

(a) Calculate the first five "Fibonacci-2" numbers.

Solution: $g_1 = 2, g_2 = 2, g_3 = 4, g_4 = 8, g_5 = 32.$

(b) Show that for all $n \in \mathbb{N}, g_n = 2^{f_n}$.

Solution: (base case) Let g_n be the Fibonacci-2 numbers defined above. Then $g_1 = 2 = 2^1$, and $g_2 = 2 = 2^1$, so $g_i = 2^{f_i}$ for $i = 1, 2$.
(induction step) Assume that $g_k = 2^{f_k}$ for all $1 \leq k < n$. We must show $g_n = 2^{f_n}$. By definition, we can write $g_n = g_{n-1}g_{n-2}$. Applying the inductive hypothesis to each factor, we have

$$g_n = g_{n-1}g_{n-2} = 2^{f_{n-1}}2^{f_{n-2}} = 2^{f_{n-1}+f_{n-2}}.$$

By the definition of the Fibonacci numbers, $2^{f_{n-1}+f_{n-2}} = 2^{f_n}$. Therefore, if the statement is true for all $1 \leq k < n$, then the statement is true for $k = n$. Thus, by the PCI, $g_n = 2^{f_n}$.

14. 'Grade' the following proofs:

(c) (see textbook for proof)
Solution: F. Claim is false. The first two Fibonacci numbers are both odd and the attempted proof does not establish a base case.

(d) (see textbook for proof)
Solution: C. Claim is true, but the attempted proof does not establish a base case and is difficult to follow/has significant gaps in the induction step.

(e) (see textbook for proof)
Solution: F. The claim is false and the attempted proof only shows one base case instead of two. Note: two base cases are necessary because the attempted proof uses the inductive hypothesis for $m - 1$ and $m - 2$.