MAT 108 Homework 15 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 1.3 #10ijk
Section 1.8 #12
Section 2.3 #9d
Section 2.4 #5i

10. . Which of the following are true in the universe of all real numbers?

(i) $(\exists! x)(\forall y)(x = y^2)$.

Solution: False. No such x exists.

(j) $(\forall y)(\exists! x)(x = y^2)$.

Solution: True. Square of a real number is well-defined.

(k) $(\exists! x)(\exists! y)(\forall w)(w^2 > x - y)$.

Solution: False. Choice of x and y is not unique. Just take $y > x$.

12. Let a be an integer and p and q be distinct primes such that p divides a and q divides a. Prove that pq divides a.

Solution: Let a be an integer and p and q be distinct primes both dividing a. Since p divides a, we can write $ps = a$ for some $s \in \mathbb{Z}$. Since q is prime and divides $a = ps$, we can apply Euclid's lemma to conclude that q divides p or q divides s. p and q were assumed to be distinct, so $q \nmid p$, from which it follows that q divides s. Therefore, we can write $qt = s$ for some $t \in \mathbb{Z}$. Together, this yields $a = pqt$. Since $t \in \mathbb{Z}$, we have shown that $pq|a$, as desired.

9. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets, $\Delta \neq \emptyset$ and B be a set. Prove the statement is true, or give a counterexample.

(d) $(\cup_{\alpha\in\Delta} A_\alpha) - B = \cup_{\alpha\in\Delta}(A_\alpha - B)$.

Solution: Let $\mathcal{A}, \Delta$, and B be given as in the statement of the problem.
($\subseteq$) Consider $x \in (\cup_{\alpha\in\Delta} A_\alpha) - B$. Then $x \in A_\alpha$ for some $\alpha \in \Delta$ and $x \notin B$. Therefore, $x \in A_\alpha - B$. Hence, $x \in \cup_{\alpha\in\Delta} A_\alpha - B$, so $(\cup_{\alpha\in\Delta} A_\alpha) - B \subseteq \cup_{\alpha\in\Delta}(A_\alpha - B)$

($\supseteq$) Let $x \in \cup_{\alpha\in\Delta}(A_\alpha - B)$. Then $x \in A_\alpha - B$ for some $\alpha \in \Delta$. By definition, $x \in A_\alpha$ and $x \notin B$. Since $x \in A_\alpha$, we know $x \in \cup_{\alpha\in\Delta} A_\alpha$. Since $x \notin B$, it follows that $x \in (\cup_{\alpha\in\Delta} A_\alpha) - B$. Therefore, $x \in \cup_{\alpha\in\Delta}(A_\alpha - B) \subseteq (\cup_{\alpha\in\Delta} A_\alpha) - B$. Thus, since we have shown both containments, the two sets are equal.

5. Use the PMI to prove the following for all natural numbers:

(i) For every prime p, for every natural number a, if p divides a^n then p divides a.

Solution: Let p be a prime number and a be any natural number
(base case) For $n = 1$, the statement becomes 'if p divides a^1, then p divides a, which is a tautology.
(induction step) Assume that $p|a^k$ implies $p|a$ for some $k \in \mathbb{N}$. We wish to show that $p|a^{k+1}$ implies $p|a$.

Consider a^{k+1} and factor it as $a^k \cdot a$. Since p is prime and divides a^{k+1}, Euclid's lemma implies that p divides a^k or p divides a. In the first case (i.e. $p|a^k$), the inductive hypothesis implies that $p|a$. Therefore, $p|a^{k+1}$ implies $p|a$.

Thus, if the statement is true for $n = k$ then the statement is true for $n = k + 1$. Hence, by the PMI, the statement is true for all $n \in \mathbb{N}$.