**MAT 108 Homework 20 Solutions**

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 4.1 #1, 2, 6bd, 18ab, 19cd

**1**. Which of the following relations are functions? For those relations that are functions, give the domain and two sets that could be a codomain.

(a) $\{(0, \triangle), (\triangle, \square), (\square, \cap), (\cap, \cup), (\cup, 0)\}$

**Solution:** Function with domain $\{0, \triangle, \square, \cap, \cup, 0\}$.
Codomain could be $\{0, \triangle, \square, \cap, \cup, 0\}$ or the set of set operations union $\{0, \square\}$.

(b) $\{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\}$

**Solution:** Not a function.

(c) $\{(1, 2), (2, 1)\}$

**Solution:** Function with domain $\{1, 2\}$. Codomain could be $\{1, 2\}$ or $\mathbb{N}$.

(d) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x = \sin y\}$

**Solution:** Not a function. This relation has $(0, 0)$ and $(0, 2\pi)$.

(e) $\{(x, y) \in \mathbb{N} \times \mathbb{N} : x \leq y\}$

**Solution:** Not a function. This relation has $(1, 1)$ and $(1, 2)$

(f) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y^2 = x\}$

**Solution:** Not a function. This relation has $(1, 1)$ and $(1, -1)$.

(g) $\{(x, y) \in \mathbb{N} \times \mathbb{N} : x^2 + y^2 < 5\}$

**Solution:** Not a function.

(h) $\{(x, y) \in \mathbb{N} \times \mathbb{N} : 2^x = 4^y\}$

**Solution:** Not a function. The domain is not $\mathbb{N}$.

(i) See textbook for diagram.

**Solution:** Not a function. The domain of the relation is not $\{a, b, c, d\}$.

(j) See textbook for diagram.

**Solution:** Function with domain $\{a, b, c, d\}$. Codomain could be $\{1, 2, 3, 4\}$ or $\mathbb{N}$.

**2**. Give two reasons why the “rule” $f(x) = \pm\sqrt{x}$ does not define a function from $\mathbb{R}$ to $\mathbb{R}$.

**Solution:** Ask in OHs or on Piazza for solution.

**6**. (b) Let $A$ be the set $\{1, 2, 3\}$, and let $R$ be the relation on $A$ given by $\{(x, y) : 3x + y \text{ is prime}\}$. Prove that $R$ is a function with domain $A$.

**Solution:** Let $A$ and $R$ be as given in the statement of the problem. We have three cases:

- When $x = 1$, we have that for $y \in A$, $3x + y$ is 4, 5, or 6. Only one of these is prime, so we have the pair $(1, 2) \in R$ and no other pair in $R$ is of the form $(1, y)$ for $y \in A$.
- When $x = 2$, we have that $3x + y$ is $7, 8$, or 9. 7 is prime, so we have $(2, 1) \in R$ and no other pair is of the form $(2, y)$ for $y \in A$ because 8 and 9 are not prime.
- When $x = 3$, we have that $3x + y$ is 10, 11, or 12. 11 is prime, so we have $(3, 2) \in R$ and no other pair in $R$ is of the form $(3, y)$ for $y \in A$ because 10 and 12 are not prime.

Therefore, $R = \{(1, 2), (2, 1), (3, 2)\}$ and satisfies $(x, y), (x, z) \in R$ implies $y = z$. Thus $R$ is a function with domain $A$.

(d) Let $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : 2x^2 - y = 1\}$. Prove that $R$ is a function with domain $\mathbb{N}$.

**Solution:** Let $R$ be given as in the statement of the problem. Suppose we have pairs $(x, y), (x, z) \in R$. Then by definition, $2x^2 - y = 1$ and $2x^2 - z = 1$. Therefore, $2x^2 - y = 2x^2 - z$. Subtracting $2x^2$ from both sides of this equation yields $y = z$. Moreover, for any $x \in \mathbb{N}$, we can solve for $y$ to get $y = 2x^2 - 1$ because $x \in \mathbb{N}$ is at least one. Hence the domain of $R$ is all of $\mathbb{N}$. Thus, we have shown that $R$ is a function on $\mathbb{N}$.

**18**. (a) Let $f$ be a function from $A$ to $B$. Define the relation $T$ on $A$ by $xTy$ iff $f(x) = f(y)$. Prove that $T$ is an equivalence relation on $A$.

**Solution:**Let $f$, $T$, $A$, $B$ be as in the statement of the problem.

- reflexive: Let $x$ in $A$. Then $f(x) = f(x)$ because $f$ is a function, so $xTX$ and $T$ is reflexive.
- symmetric: Let $x, y \in A$ such that $xTy$. Then $f(x) = f(y) = f(x)$, so $yTx$. Therefore, $T$ is symmetric.
- transitive: Let $x, y, z \in A$ such that $xTy$ and $yTz$. Then $f(x) = f(y)$ and $f(y) = f(z)$. By the transitivity of equality, $f(x) = f(z)$, so $xTz$. Therefore $T$ is transitive.

Thus, since $T$ is reflexive, symmetric and transitive, $T$ is an equivalence relation.

(b) In the case when $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$ describe the equivalence class of 0; of 2; of 4.

**Solution:** $f(0) = 0$ and no other $x \in \mathbb{R}$ satisfies $f(x) = 0$, so $\overline{0} = \{0\}$.
$f(2) = f(-2) = 4$, so $\overline{2} = \{\pm 2\}$.
$f(4) = f(-4) = 16$, so $\overline{4} = \{\pm 4\}$.

**19.** 'Grade' the following proofs:

(c) (see textbook for proof)
**Solution:** C. Proof is missing some details. The vertical line test is okay for intuition, but not a formal proof (see remarks at the top of page 206 in the text).

(d) (see textbook for proof)
**Solution:** A. Proof is good.