MAT 108 Homework 22 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 4.3 #1em, 2em, 6, 9bf, 12b, 15cfj

1. Which of the following functions map onto their codomains? Prove your answers.

(e) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{x^2 + 5}$.

Solution: Not onto/surjective. For f to be a function, we need to take $\sqrt{}$ to be positive. So $\sqrt{x^2+5} \geq 0$ and there is no $x \in \mathbb{R}$ such that $f(x) = -2$, for example.

(m) $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$, $f(\overline{x}) = \overline{2x+1}$.

Solution: Not onto/surjective. For all $\overline{x} \in \mathbb{Z}_{10}$, $f(\overline{x})$ is an equivalence class whose representatives are odd integers. This is because any two elements in the same equivalence class differ by a multiple of 10, so have the same parity, and any 2x+1 is odd for any $x \in \mathbb{Z}$.

2. Which of the functions in exercise 1 are one-to-one/injective? Prove your claims.

(e) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{x^2 + 5}$.

Solution: Not one-to-one/injective. $f(-1) = f(1) = \sqrt{6}$, but $1 \neq -1$.

(m) $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$, $f(\overline{x}) = \overline{2x+1}$

Solution: Not one-to-one/injective. We have $f(\overline{0}) = f(\overline{5}) = \overline{1}$ but $\overline{0} \neq \overline{5}$.

6. Prove that if $f : A \to B, g : B \to C$, and $g \circ f : A \xrightarrow{1-1} C$, then $f : A \xrightarrow{1-1} B$.

Solution: Let f, g be given as in the statement of the problem and assume that $g \circ f$ is one-to-one. Suppose we have some x, y in A such that $f(x) = f(y)$. We wish to show that $x = y$. Since $Dom(g) = B$, we know that $g(f(x))$ and $g(f(y))$ exist. Moreover, since g is a well-defined function, $g(f(x)) = g(f(y))$. Hence, since $g \circ f$ is one-to-one, we must have $x = y$. Thus, if $g \circ f$ is one-to-one, f is one-to-one as well.

9. Find sets A, B, C and functions $f : A \to B$ and $g : B \to C$ such that

(b) g is onto C, but $g \circ f$ is not onto C.

Solution: Let $A = B = C = \mathbb{R}$. Define $f : A \to B$ by $f(x) = e^x$ and $g : B \to C$ by $g(x) = x^3$. Then g is onto $\mathbb{R}$, but $g \circ f = (e^x)^3$ is not onto $\mathbb{R}$ because $g \circ f(x) > 0$ for all $x \in \mathbb{R}$.

(f) $g \circ f$ is one-to-one, but g is not one-to-one.
Solution: Let $A = \mathbb{R}_{\geq 0}$ and $B = C = \mathbb{R}$. Define $f : A \to B$ by $f(x) = \sqrt{x}$ and $g : B \to C$ by $g(x) = x^2$. Then $(g \circ f)(x) = x$ is one-to-one, but $g(x)$ is not one-to-one.

15. 'Grade' the following proofs:

(c) (see textbook for proof)
Solution: F. The attempted proof does not show that every element in C is in the image of $g \circ f$.

(f) (see textbook for proof)
Solution: A.

(j) (see textbook for proof)

Solution: C. The attempted proof should verify/state that each of their choices for x lie in the domain of f.