MAT 108 Homework 23 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 4.4 #1g, 2ab, 3e, 5ab, 6, 10abe

1. Show that each of these functions is a one-to-one correspondence.

(g) $f : \mathbb{Z}_8 \to \mathbb{Z}_8$ given by $f(\overline{x}) = \overline{5x-1}$

Solution: Let $\overline{x}, \overline{y} \in \mathbb{Z}_8$ and assume that $f(\overline{x}) = f(\overline{y})$. Then $\overline{5x-1} = \overline{5y-1}$, so

$$(5x-1) - (5y-1) = 8k$$

for some $k \in \mathbb{Z}$. Simplifying, we get that

$$(5x-1) - (5y-1) = 5x - 5y = 5(x-y)$$

is equal to $8k$, which tells us that $5|8k$. By Euclid's lemma, $5|k$ because 5 is prime and 5 does not divide 8. Therefore, $x - y = 8(\frac{k}{5})$ where $\frac{k}{5} \in \mathbb{Z}$. Thus, $\overline{x} = \overline{y}$ and f is one-to-one.
To show that f is a one-to-one correspondence, i.e. a bijection, take $\overline{y} \in \mathbb{Z}_8$ and consider $\overline{x} = \overline{5y+5}$. Then

$$f(\overline{x}) = \overline{5x-1} = \overline{25y+25-1} = \overline{24x+x+24} = \overline{x+8(3x+3)} = \overline{x}.$$

Thus, f is onto $\mathbb{Z}_8$, and is a one-to-one correspondence.

2. Find a one-to-one correspondence between each of these pairs of sets. Prove that your function is one-to-one and onto the given codomain.

(a) $\{a, b, c, d, e, f\}$ and $\{2, 4, 6, 8, 16, 32, 64\}$

Solution: Define the function f by $f = \{(a, 2), (b, 4), (c, 6), (d, 8), (e, 16), (f, 32)\}$. Note that f is well-defined and the range of f is contained in the codomain.
With this choice of f, we can see that $(x, y), (x', y) \in f$ imply $x = x'$, so f is one-to-one. Likewise, for any $n \in \{2, 4, 6, 8, 16, 32, 64\}$, there is a (unique) letter l in the set $\{a, b, c, d, e, f\}$ such that $f(l) = n$. Therefore, f is onto, hence bijective.

(b) $\mathbb{N}$ and $\mathbb{N} - 1$

Solution: Define the function $f(x) = x + 1$. Note that f is well defined and the range of f is contained in $\mathbb{N} - 1$.
f is injective/one-to-one: Suppose we have some $x, y \in \mathbb{N}$ such that $f(x) = f(y)$. Then $x+1 = y+1$, so $x = y$. Therefore, f is indeed one-to-one.
f is onto: Suppose we have some $b \in \mathbb{N} - 1$. Choose $a = b - 1$. Since $b \in \mathbb{N} - 1$, we know $b > 1$, so $b - 1 \in \mathbb{N}$. Then $f(a) = b - 1 + 1 = b$. Therefore, f is onto the codomain. Thus, f is a one-to-one correspondence (bijection).

3. For each one-to-one correspondence, find the inverse function. Verify your answer by computing the composite of the function and its inverse.

(e) $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ given by $f(\overline{x}) = \overline{x+2}$.

Solution: Define $f^{-1} : \mathbb{Z}_4 \to \mathbb{Z}_4$ by $f^{-1}(\overline{x}) = \overline{x-2}$ (equivalently, we could take $\overline{x+2}$. Then $f \circ f^{-1}(\overline{x}) = \overline{\overline{x-2}+2} = \overline{x}$ and similarly, $f^{-1} \circ f(\overline{x}) = \overline{\overline{x+2}-2} = \overline{x}$.

5. (a) Assume that $f : A \xrightarrow[1-1]{onto} B$ and $g : B \to A$. Prove that $g = f^{-1}$ iff $g \circ f = I_A$ or $f \circ g = I_B$.

Solution: ($\implies$) Let f, g, A, B be given as in the statement of the problem and assume that $g = f^{-1}$. By Theorem 4.2.4 (or Theorem 4.4.4a), we then have both $g \circ f = I_A$ and $f \circ g = I_B$. (Ask in OHs or piazza for a proof of Theorem 4.2.4 or see textbook)

($\impliedby$) First, assume that $g \circ f = I_A$ and consider some $a \in A$ such that $f(a) = b$. By definition, $(g \circ f)(a) = I_A(a) = a$. Moreover, $(g \circ f)(a) = g(f(a)) = g(b)$. Therefore, we have $g(b) = a$. Since f is one-to-one/injective, we know that for $a' \in A$ such that $f(a') = b$, we must have $a = a'$. Therefore, g is well-defined. Since f is onto/surjective, we know that for all $b \in B$, there is some $a \in A$ such that $f(a) = b$. Therefore, we have $Dom(g) = Dom(f^{-1}) = B$ and $g(b) = f^{-1}(b)$ for all $b \in B$. Thus, $g = f^{-1}$.

Now assume that $f \circ g = I_B$ and consider some $a \in A$ such that $f(a) = b$. We wish to show that $g(b) = a$. From the definition of the identity function on B, we have $(f \circ g)(b) = b$. We also have from the definition of composition of functions that $(f \circ g)(b) = f(g(b))$. Since f is one-to-one/injective and we have both $f(a) = b$ and $f(g(b)) = b$, we must have $a = g(b)$. Since f is onto/surjective, we also have $Dom(f^{-1}) = Dom(g) = B$. Thus, $g = f^{-1}$.

(b) Give an example of sets A and B and functions f and g such that $f : A \to B$ $g : B \to A$, $g \circ f = I_A$ and $g \neq f^{-1}$.

Solution: Let $A = \{1, 2\}, B = \{3, 4, 5\}$. Define $f : A \to B$ by $f(x) = x + 2$ and define $g : B \to A$ by $g = \{(3, 1), (4, 2), (5, 2)\}$. Then $g \circ f = id_A$ but $g \neq f^{-1}$.

6. Let $f : A \to B$ and $g : B \to A$. Prove that if $g \circ f = I_A$ and $f \circ g = I_g$, then $f : A \xrightarrow[onto]{1-1} B$ and $g : B \xrightarrow[onto]{1-1} A$.

Solution: Ask on Piazza or in OHs for solution.

12. 'Grade' the following proofs:

(a) (see textbook for proof)
Solution: F. It is not true in general that $f \circ g = g \circ f$.

(b) (see textbook for proof)
Solution: C. The computations in the attempted proof are correct, but the proof should make use of the fact that f is a permutation on A to at least state that the composite functions are defined.

(e) (see textbook for proof)
Solution: C. Should also prove that $(f \circ g)(x) = x$