MAT 108 Homework 24 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 4.4 #1cf, 2acd, 3d, 7bcd, 8, 10cd

1. Show that each of these functions is a one-to-one correspondence.

(c) $g : (-\infty, -4) \to (-\infty, 0)$ given by $g(x) = -|x + 4|$.

Solution: Let g be given as in the statement of the problem.
g is one-to-one/injective: Suppose we have some $x, x' \in (-\infty, -4)$ such that $g(x) = g(x')$. Then we have $-|x + 4| = -|x' + 4|$. Since x and x' are both less than -4, we have $x + 4 < 0$ and $x' + 4 < 0$. Therefore, $-|x+4| = x+4$ and $-|x'+4| = x'+4$. Thus, $x+4 = x'+4$, which implies $x = x'$, so g is injective.

g is onto/surjective: Take any $b \in (-\infty, 0)$ and choose $a = b+4$. Then $g(a) = -|b+4-4| = -|b|$. Since $b < 0$, we have that $-|b| = b$. Thus, g is surjective.
Hence, since g is both injective and surjective, it is bijective, i.e. g is a one-to-one correspondence.

(f) $f : [1, \infty) \to [2, \infty)$ given by $f(x) = x + \frac{1}{x}$.

Solution: Let f be given as in the statement of the problem.
f is one-to-one/injective: Suppose we have some $x, y \in [1, \infty)$ such that $f(x) = f(y)$. Then $x + \frac{1}{x} = y + \frac{1}{y}$. Clearing denominators, we can write this as $(x^2 + 1)y = x(y^2 + 1)$, or equivalently, $x(xy - 1) = y(xy - 1)$. If $xy = 1$, then $x = y = 1$. Otherwise, we can divide both sides by $(xy - 1)$ to get $x = y$. In both cases, we have shown that f is injective.
f is onto/surjective: Let $b \in [2, \infty)$ and choose $a = \frac{b+\sqrt{b^2-4}}{2}$. Then

$$f(a) = \frac{b + \sqrt{b^2 - 4}}{2} + \frac{2}{b + \sqrt{b^2 - 4}} = \frac{b^2 + 2b\sqrt{b^2 - 4} + b^2 - 4 + 4}{2(b + \sqrt{b^2 - 4})} = b.$$

Thus, f is surjective.
Hence, since f is both injective and surjective, it is bijective, i.e. f is a one-to-one correspondence.

2. Find a one-to-one correspondence between each of these pairs of sets. Prove that your function is one-to-one and onto the given codomain.

(a) $A := \{a, b, c, d, e, f\}$ and $B := \{2, 4, 8, 16, 32, 64\}$

Solution: Define the function $g : A \to B$ by $g = \{(a, 2), (b, 4), (c, 8), (d, 16), (e, 32), (f, 64)\}$. Ask in OHs or on Piazza for a proof this function is a bijection.

(c) $(3, \infty)$ and $(5, \infty)$

Solution: Define the function $f : (3, \infty) \to (5, \infty)$ by $f(x) = x + 2$. Ask in OHs or on Piazza for a proof this function is a bijection.

(d) $(-\infty, 1)$ and $(-1, \infty)$

Solution: Define the function $f : (-\infty, 1) \to (-1, \infty)$ by $f(x) = -x$. Ask in OHs or on Piazza for a proof this function is a bijection.

3. For each one-to-one correspondence, find the inverse function. Verify your answer by computing the composite of the function and its inverse.

(d) $G : (3, \infty) \to (5, \infty)$ given by $G(x) = \frac{5(x-1)}{x-3}$.

Solution: Choose $G^{-1} = \frac{3x-5}{x-5}$. Then

$$G^{-1} \circ G = \frac{3(5(x-1)/x-3) - 5}{(5(x-1)/x-3) - 5} = \frac{(3x-3)/(x-3) - 1}{(x-1)/(x-3) - 1} = \frac{2x}{2} = x.$$

Similarly,

$$G \circ G^{-1} = \frac{5((3x-5)/(x-5) - 1)}{(3x-5)/(x-5) - 3)} = \frac{5((3x-5)/(x-5) - (x-5)/(x-5))}{(3x-5)/(x-5) - 3(x-5)/(x-5)} = \frac{10x}{10} = x.$$

7. Use the one-to-one correspondences $\ln : (0, \infty) \to \mathbb{R}$ and $f : (2, \infty) \to (0, \infty)$, where $f(x) = x - 2$ to describe a one-to-one correspondence

(b) from $(2, \infty)$ onto $\mathbb{R}$

Solution: Define the composite $g : (2, \infty) \to \mathbb{R}$ by $g(x) = (\ln \circ f)(x) = \ln(x - 2)$.

(c) from $\mathbb{R}$ onto $(2, \infty)$.

Solution: Note first that $\ln^{-1}(x) = e^x$ and $f^{-1}(x) = x + 2$. Define the composite $g : \mathbb{R} \to (2, \infty)$ by $g(x) = (f^{-1} \circ \ln^{-1})(x) = e^x + 2$

(d) from $\mathbb{R}$ onto $(0, \infty)$.

Solution: Define $g(x) = \ln^{-1}(x) = e^x$.

8. Prove that if $f : A \to B, g : B \to C$ and $h : C \to A$ are one-to-one correspondences, then $f^{-1} \circ g^{-1} \circ h^{-1}$ is a permutation of A.

Solution: Let f, g, h, be as given in the statement of the problem and assume that each is a one-to-one correspondence/bijection between their domain and codomain. Since f is a bijection, f^{-1} exists and is also a bijection. Similarly, g^{-1} and h^{-1} also exist and are bijections. By Theorems 4.3.1 and 4.3.3, $g^{-1} \circ h^{-1}$ is a bijection from A to B. Again applying these theorems, the composition $f^{-1} \circ (g^{-1} \circ h^{-1})$ is a bijection from A to A, in other words, a permutation of A.

10. 'Grade' the following proofs:

(c) (see textbook for proof)
Solution: F. In general, $(r \circ s)^{-1} = s^{-1} \circ r^{-1} \neq r^{-1} \circ s^{-1}$. Here, $r \circ s \neq s^{-1} \circ r^{-1} = [42135]$.

(d) (see textbook for proof)
Solution: A.