MAT 108 Homework 25 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 5.1 #1, 4, 5, 8a, 13, 22bd

1. Prove Theorem 5.1.1. That is, show that the relation $\approx$ is reflexive, symmetric, and transitive on the class of all sets.

Solution: reflexive: Let S be a set. The identity function $id_S : S \to S$ is a bijection, so $S \approx s$. Therefore $\approx$ is reflexive.

symmetric: Let S and T be sets such that $S \approx T$. By definition, we have a bijection $f : S \to T$. Since f is a bijection, there exists a bijective inverse $f^{-1} : T \to S$. Therefore, $T \approx S$ and $\approx$ is symmetric.

transitive: Let S, T, and U be sets with $S \approx T$ and $T \approx U$. Then, there exists a bijection $f : S \to T$ and a bijection $g : T \to U$. The composition of two bijections is again a bijection, so $g \circ f : S \to U$ is a bijection and $S \approx U$. Therefore, $\approx$ is transitive.

Thus, since $\approx$ is reflexive, symmetric, and transitive, $\approx$ is an equivalence relation on sets.

4. Complete the proof that any two open intervals (a, b) and (c, d) are equivalent by showing that $f(x) = \left(\frac{d-c}{b-a}\right)(x-a)+c$ maps one-to-one and onto (c, d).

Solution: Let $f : (a, b) \to (c, d)$ be given by $f(x) = \left(\frac{d-c}{b-a}\right)(x-a)+c$.

1-1: Suppose we have some $x, y \in (a, b)$ such that $f(x) = f(y) \in (c, d)$. Then by definition

$$\left(\frac{d-c}{b-a}\right)(x-a)+c = \left(\frac{d-c}{b-a}\right)(y-a)+c.$$

Adding $-c$ to both sides, dividing by $\frac{d-c}{b-a} \neq 0$, and adding a implies that $x = y$. Therefore, f is injective/1-1.

onto: Let $x \in (c, d)$. Choose $y = \frac{b-a}{d-c}(x-c)+a$. Note that $c \neq d$, so our function is well-defined. Note further that our choice of y is in the interval (a, b) because $c \leq x \leq d$ implies $a \leq y \leq b$. With this choice of y, we have

$$f(y) = \left(\frac{d-c}{b-a}\right)(y-a)+c = \left(\frac{d-c}{b-a}\right)\left(\frac{b-a}{d-c}(x-c)+a-a\right)+c = x.$$

Thus, f is surjective/onto. Hence since f is injective and surjective, it is bijective.

5. Complete the proof of Lemma 5.1.2(b) by showing that if $h : A \to C$ and $g : B \to D$ are one-to-one correspondences, then $f : A \times B \to C \times D$ given by $f(a, b) = (h(a), g(b))$ is a one-to-one correspondence.

Solution: Let $h : A \to C$ and $g : B \to D$ be bijections and define $f : A \times B \to C \times D$ as in the statement of the problem. 1-1: Let $(a_1, b_1), (a_2, b_2) \in A \times B$ be such that $f(a_1, b_1) = f(a_2, b_2)$. Then by definition, $h(a_1) = h(a_2)$ and $g(b_1) = g(b_2)$. Since h and g are both bijections, they are, in particular, 1-1/injective, so $a_1 = a_2$ and $b_1 = b_2$. Therefore, $(a_1, b_1) = (a_2, b_2)$ and f is bijective.

onto: Let $(x, y) \in C \times D$. since h and g are both surjective, there exists some $a \in A$ such that $h(a) = x$ and some $b \in B$ such that $g(b) = y$. Choose $(a, b) \in A \times B$. Then $f(a, b) = (x, y)$ and f is surjective.

Thus, since f is both 1-1/injective and onto/surjective, f is bijective.

8. Complete the proof of Theorem 5.1.7 by proving that

(a) if A and B are finite sets, then $\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}} - \overline{\overline{A \cap B}}$.

Solution: Let A and B be finite sets. Then $B \approx \mathbb{N}_k$ for $\in \mathbb{N} \cup \{0\}$. Using our properties of sets, we can rewrite $A \cup B$ as $A \sqcup (B - A \cap B)$ where A and $B - A \cap B$ are disjoint ($\sqcup$ is just notation for disjoint union). Since A and B are both finite, their intersection is finite, so we have $A \cap B \approx \mathbb{N}_j$ for $j \leq k$. Furthermore, $B \cap A \subseteq B$ implies that $\overline{\overline{B - B \cap A}} = k - j$ by repeated application of Lemma 5.1.8.
Since A and $B - A \cap B$ are disjoint, Theorem 5.1.7(A) implies

$$\overline{\overline{A \cup B}} = \overline{\overline{A \sqcup B - A \cap B}} = \overline{\overline{A}} + \overline{\overline{B - A \cap B}} = \overline{\overline{A}} + (k - j) = \overline{\overline{A}} + \overline{\overline{B}} - \overline{\overline{A \cap B}}.$$

13. Prove that if $r > 1$ and $x \in \mathbb{N}_r$ then $\mathbb{N}_r - \{x\} \approx \mathbb{N}_{r-1}$ (Lemma 5.1.8).

Solution: Ask on Piazza for solution.

22. 'Grade' the following proofs:

(b) (see textbook for proof)
Solution: C. It's not clear from the notation that $\mathbb{N}_k$ is disjoint from $\mathbb{N}_{\{}1\}$. We require that they be disjoint because otherwise their union would be $\mathbb{N}_k$ again.

(d) (see textbook for proof)
Solution: F. The claim is false. Theorem 5.1.7 only applies to a union of a finite number of sets.