MAT 108 Homework 26 Solutions

Problems are from A Transition to Advanced Mathematics 8th edition by Smith, Eggen, and Andre.

Section 5.2 #6b, 12dg,
Section 5.3 #14abc, 16e
Section 5.4 #5, 9bc

6. (b) Give an example of a bijection h from $\mathbb{N}$ to E^+ such that $h(1) = 16$, $h(2) = 12$ and $h(3) = 2$.

Solution: Email James or ask on Piazza for solutions.

12. 'Grade' the following proofs:

(d) (see textbook for proof)
Solution: F. The claim is true, but listing the elements of an infinite set as in the attempted proof assumes that the set is countably infinite/denumerable. It is also generally good to avoid using the word 'clearly' in a proof.

(g) (see textbook for proof)
Solution: F. The claim is true, but the proof states that every subset of an infinite set is infinite, which is false. Take for example the set $\{1\} \subseteq \mathbb{N}$.

14. (a) Let S be the set of all sequences of 0's and 1's. For example, 1010101..., 101101001..., and 011111... are in S. Using a proof similar to that for Theorem 5.2.4, show that S is uncountable.

Solution: Let S be the set of all sequences of 0's and 1's. Assume towards a contradiction that S is countable. Then we can list the elements of S as $s_{11}s_{12}s_{13}\ldots, s_{21}s_{22}s_{23}\ldots, \ldots$ where $s_{ij} \in \{0, 1\}$. Consider the sequence $t = t_1t_2t_3\ldots$ defined by by

$$t_i = \begin{cases} 1 & \text{if } s_{ii} = 0 \\ 0 & \text{if } s_i i = 1 \end{cases}.$$

Then $t \in S$ but t is not in our list. This contradicts our assumption that S is countable. Thus, S must be uncountable.

(b) For each $n \in \mathbb{N}$, let T_n be the set of all sequences in S with exactly n 1's. Prove that T_n is denumerable for all $n \in \mathbb{N}$.

Solution: Let T_n be as given above. We define the function $f : T_n \to \mathbb{N}$ as follows. For a given sequence $t = t_0t_1\ldots$ in T_n, we let

$$f(t) = \sum_{i=0}^{\infty} t_i * 2^i.$$

Since $t \in T_n$, this sum has only finitely many nonzero terms and converges. Note that $2^k > \sum_{i=0}^{k-1} 2^i$. Therefore, this function is one-to-one/injective since $f(t) = f(t')$ implies that $\sum_{i=0}^{\infty} t_i * 2^i i = \sum_{i=0}^{\infty} t'_i * 2^i$ implies $t_i 2^i = t'_i 2^i$ for all i. This function is not onto, but its range is an infinite subset of the natural numbers, therefore countable by Theorem 5.3.2. Hence, if we define $\tilde{f}$ to be the restriction of f to its range, then $\tilde{f}$ is a bijection from T_n to a countable set. Thus, T_n is countable.

(c) Let $T = \cup_{k=1}^{\infty} T_k$. Use a counting process similar to that described in the discussion of Theorem 5.3.1 to show that T is denumerable.

Solution: Let $T = \cup_{k=1}^{\infty} T_k$. Consider the subset $S_k \subseteq T$ given by

$$S_k = \{t \in T : t_k = 1 \text{ and } t_j = 0 \text{ for } j > k\}.$$

Each of these sets S_k is finite (with $2^k - 1$ elements) because we can have $t_i \in \{0, 1\}$ for $i < k$. Moreover, each element in T has only a finite number of 1's, each element t must have a largest index k where $t_k = 1$ and $t_j = 0$ for $j > k$. Therefore, each element is contained in some S_k and $\cup_{k=1}^{\infty} S_k = T$. Thus, we have written T as an infinite union of finite sets and therefore T is countable by Theorem 5.3.8.

16. 'Grade' the following proofs:

(e) (see textbook for proof)

Solution: F. The claim is false. Listing elements out as in the attempted proof implicitly assumes that the set is countable and the attempted proof that f is one-to-one and onto is not at all sufficient.

5. Prove there is no largest cardinal number.

Solution: Assume towards a contradiction that there is some largest cardinal number. Call it $\aleph_k$. Then our assumption implies there is some set X with cardinality α_k and any other set X' must have cardinality less than or equal to α_k. However, Theorem 5.4.3 tells us that $\mathcal{P}(X)$ must have cardinality larger than $\aleph_k$. This contradicts our original assumption, so thus there is no largest cardinal number.

9. If possible, give an example of

(b) a one-to-one function $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$.

Solution: Not possible. $\mathcal{P}(\mathbb{N})$ is uncountable and $\mathbb{N}$ is countable, so there is no such one-to-one/injective function f.

(c) a one-to-one function $f : [4, 5] \to \mathbb{Z}$.

Solution: Not possible. $[4, 5]$ is uncountable and $\mathbb{Z}$ is countable, so there is no such one-to-one/injective function f.