Math 108 Midterm Exam February 10, 2021 1:10-2:00

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Please write and sign the following: **I pledge on my honor that I have not given or received any unauthorized assistance on this examination.**

1. (8 pts: Truth)

   (a) Construct a truth table for the proposition $\sim [(\sim P) \wedge Q]$.
   ANS: The proposition is True unless P is False and Q is True.

   (b) Find an equivalent proposition involving implies ($\implies$).
   ANS: $Q \implies P$.

2. (8 pts: Quantifiers)

   (a) $\sim [(\forall x)(\exists y) x \geq y^2]$

   (b) $(\forall x) \sim [(\forall y) x \geq y^2]$

   (c) $(\forall x)(\exists y) x < y^2$

   (d) $(\exists x)(\forall y) x < y^2$

   (a) Which pairs of the above four propositions are equivalent?
   ANS: (a) and (d) are equivalent. (b) and (c) are equivalent.

   (b) Which of the four are true in the universe $\mathbb{N}$ of natural numbers?
   ANS: (a) and (d) are False. (b) and (c) are True.

3. (8 pts: Venn)
   Prove that if A and B are sets in any universe then $A - B^c \subseteq A \cap B$.
   ANS: Proof: Assume that x is in $A - B^c$. Hence x is in A and not in B^c. Hence x is in A and in B and therefore in $A \cap B$. q.e.d.

4. (9 pts: Contrapositive)
   Prove that if n is a natural number and n^2 is not a multiple of 3 then n is not a multiple of 3.
   ANS: Proof: Assume that n is a multiple of 3. Hence there is an integer k with $3k^2$ an integer and $n = 3k$ so $n^2 = 9k^2 = 3(3k^2)$. Therefore n^2 is a multiple of 3. q.e.d.

5. (9 pts: Induction)
   Prove that if n is a natural number then 5 divides $6^n - 1$.
   ANS: Proof: Call P(n) the proposition that 5 divides $6^n - 1$. For the base case of induction consider P(1) which is that 5 divides 5 and is true. For the induction hypothesis assume that P(n) is true so 5 divides $6^n - 1$ and there is an integer k with $6^n + k$ also an integer and $6^n - 1 = 5k$. Hence $6^{n+1} - 1 = 6 \cdot 6^n - 1 = 5(6^n) + 6^n - 1 = 5(6^n + k)$ which is a multiple of 5 so P(n+1) is true. Therefore the claim holds by the principal of mathematical induction. q.e.d.

6. (8 pts: Errors)
Select the best "Proof" and find at least one serious problem with each of the others.

Claim: If $\gcd(a^2+b^2, 2ab) = 1$ then $\gcd(a,b) = 1$.

(a) "Proof": Assume $\gcd(a^2+b^2, 2ab) \neq 1$. Hence there is a prime p with $a^2+b^2 = pt$ and $2ab = pt$. Hence $(a+b)^2 = a^2+b^2+2ab = p(2t)$. Therefore p is a common divisor of a and b and $\gcd(a,b) > 1$. q.e.d.

ANS: This is a (failed) attempt to prove the (inequivalent) converse. (F)

(b) "Proof": Assume that $\gcd(a,b) \neq 1$. Hence there is $d > 1$ with $a = ds$ and $b = dt$. Hence $a^2+b^2 = d[as+bt]$ and $2ab = d(2sb)$. Therefore $\gcd(a^2+b^2, 2ab) > 1$. q.e.d.

ANS: This looks good. (A)

(c) "Proof": Assume that $a = 3$ and $b = 4$. Hence $a^2+b^2 = 9+16 = 25$ and $2ab = 24$. These have no common factors so $(25,24) = 1$. Also $(3,4) = 1$. Therefore $\gcd(a^2+b^2, 2ab) = 1$ implies $\gcd(a,b) = 1$. q.e.d.

ANS: This is a proof that there exist solutions rather than for all. (F)