Math 127B Final Spring 2025

You may use two sheets of notes.

1. Show that if f is differentiable on $(-a, a)$ and for all $-a < x < y < a$ there is $|f(x) - f(y)| \leq |x(x - y)|$ then the derivative f' is bounded on $(-a, a)$.

2. Multi-Rolle: Show that if f is thrice differentiable at every point in $(0, 5)$ and $f(1) = f(2) = f(3) = f(4)$ then there is $c \in (0, 5)$ with $f'''(c) = 0$.

3. Consider the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ with $f_n(x) = x^{\frac{1}{n}}$ defined on $[0, 1]$.

    (a) Find the pointwise limit function $f(x)$ also defined on $[0, 1]$. (Be sure to specify the value at $x = 0$.)

    (b) Show that the sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ does not converge uniformly on the entire interval $[0, 1]$.

4. Consider again the functions $f_n(x) = x^{\frac{1}{n}}$ defined on $[0, 1]$. Consider also a partition $P = \{0, a, 1\}$ of $[0, 1]$ into two intervals. Find $U(f_n; P) - L(f_n; P)$ in terms of n and a.

5. Use the fact that $\arctan(x) = \int_0^x \frac{dt}{1+t^2}$ to find the Taylor polynomial $P_{4,0}(x)$ for $\arctan(x)$.

6. The function $\phi(x)$ is smooth at every real number if

$$\phi(x) = \left\{ \begin{array}{lll} 0 & \text{if} & x \leq 0 \\ e^{\frac{-1}{x}} & \text{if} & x > 0 \end{array} \right\}.$$

   Use this to show that the function $f(x) = x\phi(x)$ is smooth at every real number.