Problem 1. Consider the zigzag function f defined on $[-1, 1]$ with $f(0) = 0$. For every positive integer n, we have:

$$f\left(\pm\frac{1}{n}\right) = \frac{(-1)^n}{n^2},$$

and f is linear on the intervals

$$\left[-\frac{1}{n}, -\frac{1}{n+1}\right] \quad \text{and} \quad \left[\frac{1}{n+1}, \frac{1}{n}\right].$$

At which points in $(-1, 1)$ is f differentiable?

Solution. It is clear from the construction of f that f is differentiable on $(-\frac{1}{n}, -\frac{1}{n+1})$ and not differentiable at $x = -\frac{1}{n}, \frac{1}{n}$, for $n = 1, 2, ...$ We will show that $f'(0) = 0$. Fix $\epsilon > 0$. Let N be a positive integer such that $\frac{n+1}{n^2} < \epsilon$ for all $n \geq N$. Next, choose $\delta > 0$ be sufficiently small such that for all $x \in (-\delta, \delta)\backslash\{0\}$, there exists $n \geq N$ such that $x \in [-\frac{1}{n}, -\frac{1}{n+1}]$ or $x \in [\frac{1}{n+1}, \frac{1}{n}]$. Then we have for all x such that $|x| < \delta$ and $x \neq 0$,

$$|\frac{f(x)}{x}| \leq \frac{\frac{1}{n^2}}{\frac{1}{n+1}} = \frac{n+1}{n^2} < \epsilon.$$

Problem 2. Show that if f is twice differentiable in (a, b) and $a < c < b$, then

$$f''(c) = \lim_{h\to 0} h^{-2}\left[f(c-h) - 2f(c) + f(c+h)\right].$$

Solution.

Problem 3. Assume that f is twice differentiable at every point in $(0, 4)$, $f(3) = 3$, $f(1) = 0$, and there is some other number $x \in (1, 3)$ with $f(x) = 0$ also. Show that there is some number $y \in (1, 3)$ with $f''(y) > \frac{3}{4}$.

Solution. Let $x_0 \in (1, 3)$ such that $f(x_0) = 0$. By MVT, there exists $a \in (1, x_0)$ such that $f'(a) = \frac{f(x_0)-f(1)}{x_0-1} = 0$. Similarly, there exists $b \in (x_0, 3)$ such that $f'(b) = \frac{f(3)-f(x_0)}{3-x_0} = \frac{3}{3-x_0}$. Since $x_0 \in (1, 3)$, $f'(b) > \frac{3}{2}$. Finally, applying MVT to f' on $[a, b]$, there exists $c \in (a, b)$ such that $f''(c) = \frac{f'(b)-f'(a)}{b-a} = \frac{f'(b)}{b-a} > \frac{\frac{3}{2}}{2} = \frac{3}{4}$.

Problem 4. Consider the sequence of functions $\{f_n\}$ defined on $[0, \infty)$ with

$$f_n(x) = \frac{nx}{1+nx}.$$

(a) Find the pointwise limit of this sequence. (Pay attention to zero.)
(b) Determine whether the sequence converges uniformly.

Solution.

(a) We have $f_n(0) = 0$ for all n, and for $x \in (0, \infty)$, $f^n(x) \to 1$ as $n \to \infty$. Therefore, f_n converge to g pointwise, where

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

(b) The convergence is not uniform because for any interger $n > 0$, $f(\frac{1}{n}) = \frac{1}{2}$.

Problem 5. Consider the two equal parts partition $P = \left\{0, \frac{1}{2}, 1\right\}$ of $[0, 1]$. Show that there is another two part partition Q of $[0, 1]$ so that

$$U(x^2, Q) - L(x^2; Q) < U(x^2, P) - L(x^2; P).$$

Solution. For this problem, you just need to try something. For example, try $Q = \{0, \frac{2}{3}, 1\}$.

Problem 6. Assume that f is integrable on $[0, 4]$ and define

$$F(x) = \int_0^x f$$

also on $[0, 4]$.

Show that there is some $c \in [2, 3]$ with

$$\int_2^3 F = \int_0^c f.$$

Solution. The function $F(x) = \int_0^x f(t)dt$ is continuous on $[0, 4]$ since f is integrable on $[0, 4]$. Therefore, $h(x) = \int_0^x F(t)dt$ is integrable on $(0, 4)$, by the Fundamental Theorem of Calculus. By MVT, there exists $c \in [2, 3]$ such that $h'(c) = \frac{h(3)-h(2)}{3-2} = \int_2^3 F$. Also by Fundamental Theorem of Calculus, we have $h'(c) = F(c) = \int_0^c f$.

Problem 7. Show that if $\int_I f^2 = 0$ then $\int_I f$ exists.

Solution. Since f^2 is integrable, f^2 continuous almost everywhere on I (i.e., the set of discontinuities of f^2 in I has measure zero). In addition, since $\int_I f^2 = 0$, $f^2(x) = 0$ for all x at which f is continuous. This implies that $f = 0$ almost everywhere on I (i.e., the set of points of I where f is nonzero has measure zero.) Therefore, f is integrable. To prove this, fix $\epsilon > 0$ and choose a finite collection of disjoints intervals I_k that cover the set of points where f is nonzero such that $m(\bigcup I_k < \epsilon)$. Refine these intervals to make a partition P for I. Then we can see without difficulty that $U(f, P) - L(f, P) < C\epsilon$, where $C = 2\sup_{x \in I} |f(x)|$. (Note, f is bounded.)

Problem 8. Show that if $f(0) = 0$ and $f(x) = P_{\infty,0}(x)$ (its Taylor series) on $(-a, a)$ then so does $g(x)$ with $g(0) = f'(0)$ and $g(x) = x^{-1}f(x)$ otherwise.

Solution. By assumption, $f(0) = 0$, and so $f(x) = \sum_{n=1}^{\infty} a_n x^n$, where $a_n = f^{(n)}(0)/n!$. Thus, when $x \neq 0$, we have $\frac{f(x)}{x} = \sum_{n=1}^{\infty} a_n x^{n-1}$.Define

$$\tilde{g}(x) = \begin{cases} f'(0) & \text{if } x = 0 \\ \sum_{n=1}^{\infty} a_n x^{n-1} & \text{if } x \neq 0. \end{cases}$$

Then $\tilde{g}(x) = g(x)$ for all $x \in (-a, a)$. Moreover, $\lim_{x\to 0} \sum_{n=1}^{\infty} a_n x^{n-1} = f'(0)$, thus we can write $\tilde{g}(x) = \sum_{n=0}^{\infty} a_{n+1} x^n$ for all $x \in (-a, a)$. Note that $\tilde{g}$ and g has the same radius of convergence.

Problem 9. hehe

Solution. We have $f'(x) = \ln(1 + x), f''(x) = \frac{1}{1+x}, f'''(x) = -\frac{1}{(1+x)^2}$. Thus,

$$P_{2,0}(x) = \frac{1}{2}x^2.$$

We have $R_2(x) = f(x) - P_{2,0}(x) = \frac{-1}{6(1+\xi)^2}x^3$ for some ξ between 0 and x. If $x > 0$ and $x < 1$, then $0 < \xi < x < 1$. Thus $-\frac{x^3}{6} > R_2(x) > -\frac{x^3}{24}$. If $-1 < x < 0$, then $-1 < x < \xi < 0$, and so $(1 + \xi)^2 \in ((x + 1)^2, 1)$. Therefore, $-\frac{x^3}{6(1+x)^2} < R_2(x) < -\frac{x^3}{6}$.