Math 127B Midterm I Spring 2025

Solutions

Problem: 1. Derivative

Assume that f and g are functions defined on all reals with:

(a) $f(0) = g(0) = 0$,

(b) $(\forall x \in \mathbb{R})\ f(x) \geq |g(x)|$,

(c) $g'(0) = 1$.

Show that $f'(0)$ does not exist.

In terms of limits, $f'(0)$ (if it were to exist) and $g'(0)$ say:

$$f'(0) = \lim_{x \to 0} \frac{f(x)}{x}$$
$$1 = \lim_{x \to 0} \frac{g(x)}{x}$$

The key for showing that $f'(0)$ does not exist is to realize $x \to 0$ from both sides in these limits. On the left-hand side $x \to 0^-$ with $x < 0$ and on the right-hand side $x \to 0^+$ with $x > 0$, and x is the denominator in both limits above. $g'(0)$ is positive, so $g(x) < 0$ for $x < 0$ and $g(x) > 0$ for $x > 0$, while $f(x) \geq |g(x)| \geq 0$.

Now to be a little more specific. Unwinding the statement $g'(0) = 1$ says that for any $\varepsilon > 0$ you can find a $\delta > 0$ such that when $0 < |x| < \delta$ we have that:

$$\left| \frac{g(x)}{x} - 1 \right| < \varepsilon$$
$$-\varepsilon < \frac{g(x)}{x} - 1 < \varepsilon$$
$$1 - \varepsilon < \frac{g(x)}{x} < 1 + \varepsilon$$

In particular, we can choose (somewhat arbitrarily, just to introduce concrete numbers into the problem) $\varepsilon = 1/2$ and then:

$$\frac{1}{2} < \frac{g(x)}{x} < \frac{3}{2}$$

For $x > 0$ (now using absolute value bars on x to keep things simpler):

$$\frac{|x|}{2} < g(x) < \frac{3|x|}{2}$$

and for $x < 0$:

$$-\frac{3|x|}{2} < g(x) < -\frac{|x|}{2}$$

Expanding out the fact that $f(x) \geq |g(x)|$ for both cases, first for $x > 0$:

$$f(x) > \frac{|x|}{2}$$

and for $x < 0$:

$$f(x) > \frac{|x|}{2}$$

So for $x > 0$:

$$\begin{aligned}\frac{f(x)}{x} &> \frac{|x|/2}{x} \\ &= \frac{1}{2}\end{aligned}$$

and for $x < 0$:

$$\begin{aligned}\frac{f(x)}{x} &< \frac{|x|/2}{x} \\ &= \frac{|x|/2}{-|x|} \\ &= -\frac{1}{2}\end{aligned}$$

So if $f'(0^+)$ were to possibly exist, it would be at most $-1/2$ and if $f'(0^+)$ were to exist it would be at least $1/2$. These numbers are different, so $f'(0)$ cannot possibly exist.

Problem: 2. Mean Mean Value

Assume that h is a function defined on all reals with:

(a) $h(0) = 0$,

(b) h is even [that is $(\forall b)\ h(-b) = h(b)$],

(c) h is twice differentiable [that is $(\forall b)\ h''(b)$ exists].

Show that for every b there is c with $b^2 h''(c) \geq h(b)$.

First a preliminary result: since h is even, h' is odd $(h'(-b) = -h'(b))$. To see this:

$$\begin{aligned} h'(b) &= \lim_{\Delta b \to 0} \frac{h(b + \Delta b) - h(b)}{\Delta b} \\ &= \lim_{\Delta b \to 0} \frac{h(-b - \Delta b) - h(-b)}{\Delta b} \\ &= \lim_{\Delta b' \to 0} \frac{h(-b + \Delta b') - h(-b)}{-\Delta b'} \\ &= -h'(-b) \end{aligned}$$

where $\Delta b' = -\Delta b$. This also means $h'(0) = 0$ (since $h'(0) = -h'(-0) = h'(0)$).

Now for the proof. The case $b = 0$ is automatic. If we can show this is true for $b > 0$, then it holds for $b < 0$ since:

$$\begin{aligned} h(b) &= h(-b) \\ &\leq (-b)^2 h''(c) \\ &= b^2 h''(c) \end{aligned}$$

So take a $b > 0$. The mean value theorem on h says there is a d between 0 and b such that:

$$h'(d) = \frac{h(b)}{b}$$

The mean value theorem on h' says that there is a c between 0 and d such that:

$$\begin{aligned} h''(c) &= \frac{h'(d) - h'(0)}{d - 0} \\ &= \frac{h'(d)}{d} \\ &= \frac{h(b)}{bd} \\ \rightsquigarrow b^2 h''(c) &= \frac{b}{d} h(b) \\ &\geq h(b) \end{aligned}$$

where in this last line I am assuming $h(b) \geq 0$ and using the fact that $b/d \geq 1$.

If $h(b) < 0$, the trick is to find a c where $h''(c) \geq 0$, since in that case:

$$\begin{aligned} b^2 h''(c) &\geq 0 \\ &> h(b) \end{aligned}$$

The mean value theorem on h says there is a d between $-b$ and b such that:

$$\begin{aligned} h'(d) &= \frac{h(b) - h(-b)}{b - (-b)} \\ &= 0 \end{aligned}$$

The mean value theorem on h' says that there is a c between 0 and d (here I am writing it like $d \geq 0$, but if $d < 0$ the numerator is $-$ what I write and $-0 = 0$) such that:

$$\begin{aligned} h''(c) &= \frac{h'(d) - h'(0)}{d - 0} \\ &= \frac{0 - 0}{d} \\ &= 0 \end{aligned}$$

So I have found a c where $h''(c) \geq 0$, which I said is sufficient for the $h(b) < 0$ case.

Problem: 3. Convergence

Show that the sequence of functions

$$\left\{ b_n(x) = n \sin\left(\frac{x}{n}\right) \right\}$$

(a) converges pointwise on all reals to the function x but

(b) does not converge uniformly on all reals to the function x.

Hint: Consider using L'Hospital's rule in (a).

(a) Computing the pointwise limit of $b_n(x)$ as $n \to \infty$:

$$\begin{aligned}
\lim_{n\to\infty} b_n(x) &= \lim_{n\to\infty} n \sin\left(\frac{x}{n}\right) \\
&= \lim_{n\to\infty} \frac{\sin(x/n)}{1/n} \\
&= \lim_{n\to\infty} \frac{(-x/n^2)\cos(x/n)}{-1/n^2} \qquad (***) \\
&= \lim_{n\to\infty} x \cos\left(\frac{x}{n}\right) \\
&= x
\end{aligned}$$

In going to the line labeled with (***), L'Hospital's rule was used in the 0/0 indeterminate form. This shows that $b_n(x) \to x$ pointwise for any x.

(b) If b_n were to converge on $\mathbb{R}$ to x, then for any $\varepsilon > 0$ we could find a natural number N (possibly dependent on ε, but definitely not x) such that when $n > N$ we have that:

$$\left| n \sin\left(\frac{x}{n}\right) - x \right| < \varepsilon$$

Take $x = n\pi$. Then we would be assured that:

$$\begin{aligned}
|n \sin(\pi) - n\pi| &= n\pi \\
&< \varepsilon
\end{aligned}$$

but this is not true.

Problem: 4. Series

Show that the sequence of functions

$$\left\{k_n = \sum_{t=1}^{n}\left(\frac{\sin(xt)}{t^3}\right)\right\}$$

converges pointwise on all reals to a differentiable function.

Hint: Feel free to use the fact that

$$\sum_{t=M+1}^{\infty}\frac{1}{t^2} \leq \int_M^{\infty}\frac{dx}{x^2} = \frac{1}{M}$$

I will solve this problem similar to problem 4 on the practice midterm, by using the same theorem from Hunter copied again here for reference:

Theorem 9.18. Suppose that (f_n) is a sequence of differentiable functions $f_n : (a,b) \to \mathbb{R}$ such that $f_n \to f$ pointwise and $f_n' \to g$ uniformly for some $f, g : (a,b) \to \mathbb{R}$. Then f is differentiable on (a,b) and $f' = g$.

Denote $g_t(x) = \sin(xt)/t^3$. Then $k_n(x) = \sum_{t=1}^{n} g_t(x)$. I will use the Weierstrass test to show that k_n converges uniformly to some function k. First to find some Majorants M_t to use for this test:

$$\begin{aligned} |g_t(x)| &= \left|\frac{\sin(xt)}{t^3}\right| \\ &\leq \frac{1}{t^3} \\ &=: M_t \end{aligned}$$

Now:

$$\begin{aligned} \sum_{t=1}^{\infty} M_t &= S + \sum_{t=M+1}^{\infty}\frac{1}{t^3} \\ &\leq S + \sum_{t=M+1}^{\infty}\frac{1}{t^2} \\ &\leq S + \frac{1}{M} \end{aligned}$$

where $S = \sum_{t=1}^{m} M_t$ is a finite number for some fixed natural number M. So $\sum_{t=1}^{\infty} M_t$ is finite. So by the Weierstrass test $k_n \to k$ uniformly for some function k (in particular $k_n \to k$ pointwise as well).

Now taking a derivative:

$$\begin{aligned} k_n'(x) &= \sum_{t=1}^{n} g_t'(x) \\ &= \sum_{t=1}^{n} \frac{\cos(xt)}{t^2} \end{aligned}$$

Again using the Weierstrass test, by first finding some Majorants N_t:

$$\begin{aligned} |g_t'(x)| &= \left| \frac{\cos(xt)}{t^2} \right| \\ &\leq \frac{1}{t^2} \\ &=: N_t \end{aligned}$$

Like before the sum of all N_t's is finite:

$$\begin{aligned} \sum_{t=1}^{\infty} N_t &= T + \sum_{t=N+1}^{\infty} \frac{1}{t^3} \\ &\leq T + \frac{1}{N} \end{aligned}$$

for some natural number N and finite number T. So $k_n' \to g$ uniformly for some function g.

By the cited theorem, f is differentiable.