Math 127B Practice Midterm II Spring 2025

You may use one sheet of notes.

Your score will be the best 4 of 5.

1. (power series) Write A, B and C for the radii of convergence of the three power series $\sum_{k=0}^{\infty} a_k x^k$, $\sum_{k=0}^{\infty} b_k x^k$ and $\sum_{k=0}^{\infty} a_k b_k x^k$.

   (a) Show that $C \geq AB$.

   (b) Find an example with $C = AB$.

   (c) Find an example with $C > AB$.

2. (partition) Consider the topologist's sine curve with $f(x) = \sin(\frac{1}{x})$ if $0 < |x| \leq 1$ and $f(0) = 0$. Find a partition P of $[-1, 1]$ for which $U(f, P) - L(f, P) < 1$.

3. (integrable algebra) Consider a bounded function f on the interval $I = [a, b]$. Write $f_+(x) = f(x)$ if $f(x) \geq 0$ and $f_+(x) = 0$ otherwise. Write $f_-(x) = f_+(x) - f(x)$. Thus $f_\pm$ are both non-negative and $f = f_+ - f_-$. Show that f is integrable on I if and only if f_+ and f_- are both integrable on I.

4. (ftc) Assume that f is an integrable function on $I = [0, 1]$ and hence that $F(x) = \int_0^x f$ is defined on I.

   (a) Show that F is differentiable at some point in $(0, 1)$.

   (b) Find an example for which F is not differentiable at $\frac{1}{2}$.

5. (discontinuity) Find a bounded function f on $[-1, 1]$ for which $D^1(f) = \{\frac{1}{2}\} \subseteq D^{\frac{1}{2}}(f) = [-1, 0] \cup \{\frac{1}{2}\} \subseteq D(f) = [-1, 0] \cup \{\frac{1}{2}, 1\}$ and $\int_0^1 f = 1$.