Math 127B Practice Midterm II Solutions Spring 2025

You may use one sheet of notes.

Your score will be the best 4 of 5.

1. (power series) Write A, B and C for the radii of convergence of the three power series $\sum_{k=0}^{\infty} a_k x^k$, $\sum_{k=0}^{\infty} b_k x^k$ and $\sum_{k=0}^{\infty} a_k b_k x^k$.

   (a) Show that $C \geq AB$.

   (b) Find an example with $C = AB$.

   (c) Find an example with $C > AB$.

   ANS:

   (a) $A^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ and $B^{-1} = \limsup_{n\to\infty} |b_n|^{\frac{1}{n}}$ so $A^{-1}B^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \limsup_{n\to\infty} |b_n|^{\frac{1}{n}} =$ $\lim_{n\to\infty} \sup_{k>n} |a_n|^{\frac{1}{n}} \sup_{k>n} |b_n|^{\frac{1}{n}} \geq$ $\lim_{n\to\infty} \sup_{k>n} |a_n b_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |a_n b_n|^{\frac{1}{n}} = C^{-1}$

   (b) Take $a_n = b_n = 1$ with $A = B = C = 1$.

   (c) Take $a_{2n} = b_{2n+1} = 1$ and $a_{2n+1} = b_{2n} = 0$ with $A = B = 1$ and $C = \infty$.

2. (partition) Consider the topologist's sine curve with $f(x) = \sin(\frac{1}{x})$ if $0 < |x| \leq 1$ and $f(0) = 0$. Find a partition P of $[-1, 1]$ for which $U(f, P) - L(f, P) < 1$.

   ANS: Take $P = \{\frac{a}{256} | -256 \leq a \leq 256\}$. Write $I_k = [\frac{k-1}{256}, \frac{k}{256}]$ and if $k \geq 32$ or $k < -32$ then f is differentiable on the interior of I_k and the supremum of the absolute value of the derivative is at most 64 so by the MVT $|M_{I_k} f - m_{I_k} f| \leq \frac{64}{256} = \frac{1}{4}$. Finally $M_{[-1,1]}|f| \leq 2$. Combining these $U(f, P) - L(f, P) = \sum_{k=-255}^{256} (M_{I_k} f - m_{I_k} f) \frac{1}{256} < 2\frac{1}{4} + \frac{1}{4}2$ where the first term is

$M_I|f| = 2$ times the total width of the intervals I_{-32} through I_{31} and the second is the maximum oscillation in the remaining intervals times the total length of $I = [-1, 1]$.

3. (integrable algebra) Consider a bounded function f on the interval $I = [a, b]$. Write $f_+(x) = f(x)$ if $f(x) \geq 0$ and $f_+(x) = 0$ otherwise. Write $f_-(x) = f_+(x) - f(x)$. Thus $f_\pm$ are both non-negative and $f = f_+ - f_-$. Show that f is integrable on I if and only if f_+ and f_- are both integrable on I.

   ANS: Since $f = f_+ + f_-$ if $f_\pm$ are both integrable then so is f. Since $f_\pm = \frac{1}{2}(|f| \pm f)$ if f is integrable then so is $|f|$ and hence both $f_\pm$ are also.

4. (ftc) Assume that f is an integrable function on $I = [0, 1]$ and hence that $F(x) = \int_0^x f$ is defined on I.

   (a) Show that F is differentiable at some point in $(0, 1)$.

   (b) Find an example for which F is not differentiable at $\frac{1}{2}$.

   ANS: For (a) every integrable function is continuous at at least one point c and hence by the FTC $F'(c) = f(c)$ exists. For (b) take $f = \chi_{[\frac{1}{2},1]}$ and compute that $F(x) = 0$ if $x \leq \frac{1}{2}$ and $F(x) = x - \frac{1}{2}$ otherwise which is not differentiable (it has a corner) at $x = \frac{1}{2}$.

5. (discontinuity) Find a bounded function f on $[-1, 1]$ for which $D^1(f) = \{\frac{1}{2}\} \subseteq D^{\frac{1}{2}}(f) = [-1, 0] \cup \{\frac{1}{2}\} \subseteq D(f) = [-1, 0] \cup \{\frac{1}{2}, 1\}$ and $\int_0^1 f = 1$.

   ANS: Take $f(\frac{1}{2}) = 3$, $f(1) = \frac{3}{4}$, $f(x) = \frac{1}{4}$ if x is negative and rational, and $f(x) = 1$ otherwise.