MAT 127B Midterm 2 solutions

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1. Find
$$\sum_{n=1}^{\infty} \frac{1}{n2^n}.$$

Solution:

Start with the geometric series
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Integration gives
$$\int_0^x \frac{1}{1-t}\,dt = \int_0^x \sum_{n=0}^{\infty} t^n\,dt$$

As the sum converges uniformly, we can pass the integrals through the sum and get
$$-\ln|1-t| = \sum_{n=0}^{\infty} \int_0^x t^n\,dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

so putting $x = 1/2$ (which is within the radius of convergence),
$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = -\ln|1-1/2| = -\ln 2^{-1} = \ln 2$$

2. Find a function f with
$$U(f;\{0,1\}) - L(f;\{0,1\}) = 1$$

and
$$U\left(f;\left\{0,\tfrac{1}{2},1\right\}\right) - L\left(f;\left\{0,\tfrac{1}{2},1\right\}\right) = \frac{1}{2}.$$

Solution:

Let $f : [0,1] \to \mathbb{R}$ be the function
$$f(x) = \chi_0(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

One checks easily that
$$U(f;\{0,1\}) = 1, \quad L(f;\{0,1\}) = 0$$
$$U\left(f;\left\{0,\tfrac{1}{2},1\right\}\right) = \frac{1}{2}, \quad L\left(f;\left\{0,\tfrac{1}{2},1\right\}\right) = 0$$

3. Show that $\int_0^1 \chi_{1/2} = 0$ by finding for every $\epsilon > 0$ a partition P of $[0,1]$ with

$$-\epsilon < L(\chi_{1/2}; P) < U(\chi_{1/2}; P) < \epsilon.$$

Here $\chi_{1/2}$ is the characteristic function defined by $\chi_{1/2}(\frac{1}{2}) = 1$ and if $x \in [0,1] \setminus \{\frac{1}{2}\}$ then $\chi_{1/2}(x) = 0$. Your P will depend on ϵ.

Solution:

Let

$$P = \left\{0, \frac{1}{2} - \frac{\epsilon}{3}, \frac{1}{2} + \frac{\epsilon}{3}, 1\right\}$$

Then

$$L(\chi_{1/2}; P) = \left(\frac{1}{2} - \frac{\epsilon}{3}\right) \cdot 0 + \frac{2\epsilon}{3} \cdot 0 + \left(\frac{1}{2} - \frac{\epsilon}{3}\right) \cdot 0 = 0$$

$$U(\chi_{1/2}; P) = \left(\frac{1}{2} - \frac{\epsilon}{3}\right) \cdot 0 + \frac{2\epsilon}{3} \cdot 1 + \left(\frac{1}{2} - \frac{\epsilon}{3}\right) \cdot 0 = \frac{2\epsilon}{3}$$

and therefore $-\epsilon < L(\chi_{1/2}; P) < U(\chi_{1/2}; P) < \epsilon$.

4. Assume that f and g are bounded functions on $[0,1]$.

   (a) Show that if f and g are continuous at c then so is $f + g$.

   (b) Show that $D(f+g) \subseteq D(f) \cup D(g)$.
   Here $D(f)$ is the set of values in $[0,1]$ at which f is not continuous.

Solution:

(a) This is clear from the equation

$$\lim_{h \to 0} f(c+h) + g(c+h) = f(c) + g(c)$$

since f, g are continuous at c. You can also do it more formally. Let $\epsilon > 0$. Since by continuity, there is $\delta > 0$ such that for all $|h| < \delta$,

$$|f(c+h) - f(c)| < \frac{\epsilon}{2}$$

$$|g(c+h) - g(c)| < \frac{\epsilon}{2}$$

By triangular inequality,

$$|(f(c+h)+g(c+h)) - (f(c)+g(c))| \leq |f(c+h)-f(c)| + |g(c+h)-g(c)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(b) By (a), we have

$$D(f)^c \cap D(g)^c \subseteq D(f+g)^c$$

Taking complement on both sides yields

$$D(f) \cup D(g) \supseteq D(f+g)$$

5. Assume that f is monotone increasing from 0 to 1 on $[0,1]$ (that is: if $0 \le x \le y \le 1$ then $0 = f(0) \le f(x) \le f(y) \le f(1) = 1$). Show that $\lim_{n\to\infty} \int f^n$ exists.

   Solution:

   We first prove this for the case $f(x) \neq 1$ for all $x \neq 1$, and the general case follows quickly from this.

   In this case, we claim that the limit is 0. First, observe that

$$f(x) := \lim_{n\to\infty} f(x)^n = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

   If f is continuous (from the left) at $x = 1$. Then $f_n \to f$ pointwisely but not uniformly. (There is no f^n strictly within a ϵ-tubular neighborhood of f, due to continuity.) Thus, we can't pass the limit into the integral in general. Instead, we shall prove it using the definition of limit. Let $\epsilon > 0$ and consider the point $1 - \frac{\epsilon}{2}$. We can split the integral into

$$\int_0^1 f^n = \int_0^{1-\frac{\epsilon}{2}} f^n + \int_{1-\frac{\epsilon}{2}}^1 f^n$$

   Now let $N \in \mathbb{N}$ sufficiently large so that

$$f(1 - \frac{\epsilon}{2})^N < \frac{\frac{\epsilon}{2}}{1 - \frac{\epsilon}{2}}$$

   Since f is monotonically increasing, we have for all $n > N$,

$$\int_0^{1-\frac{\epsilon}{2}} f^n + \int_{1-\frac{\epsilon}{2}}^1 f^n < \left(1 - \frac{\epsilon}{2}\right) f(1 - \frac{\epsilon}{2})^n + \frac{\epsilon}{2} f(1) < \epsilon$$

   so $\lim_{n\to\infty} \int_0^1 f^n = 0$.

   For the general case, let

$$c = \inf\{x \in [0,1] \mid f(x) = 1\}$$

   so that c is the smallest possible value such that $f(x) < 1$ for all $x < c$. Then

$$\int_0^1 f^n = \int_0^c f^n + \int_c^1 f^n = \left(\int_0^c f^n\right) + (1 - c)$$

   By the previous case, $\lim_{n\to\infty} \int_0^c f^n = 0$, so $\lim_{n\to\infty} \int_0^1 f^n$ exists.