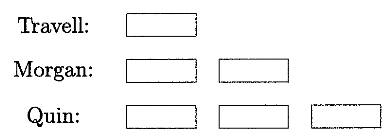
— Exercises —

Exercise 8.1. Travell, Morgan and Quin are splitting an Uber to get to their homes; they all live on the same straight street, which is also the same street they are getting picked up on. After driving 10 miles on this street they reach Travell's house and he gets out, after another 10 miles they reach Morgan's house and she gets out, after another 10 miles they reach Quin's house and he gets out.

The cost of the Uber is \$1.50 per mile, so in total they have to pay \$45. They discuss how much should each person have to pay for it to be fair. They agree that since their rides were different distances, they should pay different amounts, but Travell and Quin come up with different ways to divide the cost. Travell sends his reasoning, and it looks like this:

$$\begin{cases} \text{Travell} & = \frac{1}{3}\cdot 15 = \$5 \\ \text{Morgan} & = \frac{1}{3}\cdot 15 + \frac{1}{2}\cdot 15 = \$5 + \$7.50 = \$12.50 \\ \text{Quin} & = \frac{1}{3}\cdot 15 + \frac{1}{2}\cdot 15 + 15 = \$5 + \$7.50 + \$15 = \$27.50 \end{cases}$$

But Quin sends his reasoning and it looks like this:



$$\Longrightarrow \quad \begin{cases} \text{Travell} & = \frac{1}{6}\cdot 45 = \$7.50 \\ \text{Morgan} & = \frac{2}{6}\cdot 45 = \$15 \\ \text{Quin} & = \frac{3}{6}\cdot 45 = \$22.50 \end{cases}$$

Explain each person's reasoning. Why do they get different answers? Who do you think is correct? Explain why theirs is better than the other.

Exercise 8.2. Prove that if A and B are nonempty bounded sets and $A \subseteq B$, then $\sup(A) \leq \sup(B)$ and $\inf(B) \leq \inf(A)$.

Exercise 8.3. Suppose A and B are nonempty sets of real numbers such that for any $x \in A$ and $y \in B$ we have $x \leq y$. Prove that $\sup(A) \leq \inf(B)$.

Exercise 8.4. Let $f : [0,2] \to \mathbb{R}$ be defined by $f(x) = x^2 - x$, and let $P = \{0, 1, \frac{3}{2}, 2\}$. Compute $U(f,P)$ and $L(f,P)$ with respect to this particular partition P.

Exercise 8.5. Suppose $a \leq b < c \leq d$ and that f is integrable on $[a,d]$. Prove that f is also integrable on $[b,c]$.

Exercise 8.6. Give an example of a function $f : [0,1] \to \mathbb{R}$ for which f is *not* integrable, but $(f(x))^2$ *is* integrable.

Exercise 8.7. Recall that the *modified Dirichlet function* is defined to be

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Let P be a partition of $[0,4]$. Compute $L(g,P)$.

(b) Find $\inf\{U(g,P) : P \text{ a partition of } [0,4]\}$.

Exercise 8.8. Prove Corollary 8.15. That is, prove that if $f : [a,b] \to \mathbb{R}$ is integrable, then there exists a sequence P_n of partitions of $[a,b]$ for which

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0.$$

Exercise 8.9. In this exercise you do not need to prove that your example works. Give an example of a function $f : [0,2] \to \mathbb{R}$ which has the following two properties:

- When restricted to the domain $[0,1]$ (so now $f : [0,1] \to \mathbb{R}$), we have

$$L(f,P) < U(f,P)$$

for every partition P of $[0,1]$.

- But when restricted to the domain $[1,2]$ (so now $f : [1,2] \to \mathbb{R}$), we have

$$L(f,P) = U(f,P)$$

for every partition P of $[1,2]$.

Exercise 8.10. Give an example of numbers a and b, and of integrable functions $f, g : [a,b] \to \mathbb{R}$, where

$$U(f+g,P) < U(f,P) + U(g,P)$$

for every partition P of $[a,b]$. Make sure to prove your answer.

Exercise 8.11.

(a) Prove that $\displaystyle\int_0^b x\,dx = \frac{b^2}{2}$ by considering partitions into n equal subintervals.

(b) Prove that $\displaystyle\int_0^b x^2\,dx = \frac{b^3}{3}$ by considering partitions into n equal subintervals.

(c) Prove that $\displaystyle\int_0^b x^3\,dx = \frac{b^4}{4}$ by considering partitions into n equal subintervals.

Exercise 8.12. Consider the function $f : [0,3] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 2 & \text{if } x \in [1,2) \\ 3 & \text{if } x \in [2,3] \end{cases}$$

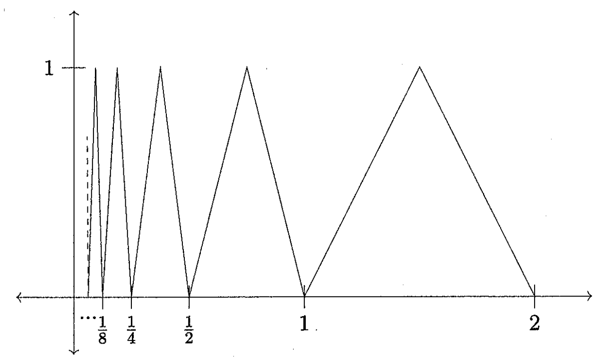
Using the integrals analytically theorem, prove that f is integrable.

Exercise 8.13. Consider the function $s : [0,2] \to \mathbb{R}$ given by

$$s(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 5 & \text{if } x = 1 \\ 2 & \text{if } x \in (1,2] \end{cases}$$

Using the integrals analytically theorem, prove that s is integrable.

Exercise 8.14. Let f be the function graphed below, where the zig-zag pattern continues, and where $f(0) = 0$. Is f integrable on $[0,2]$?



Exercise 8.15. Suppose f and g are integrable on $[a,b]$.

(a) Prove that if there exists some $c \in [a,b]$ such that $f(x) = g(x)$ for all $x \neq c$, then $\displaystyle\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.

(b) If $f(x) = g(x)$ for all but countably many x values in $[a,b]$, must it be the case that $\displaystyle\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$?

Exercise 8.16. A function f is *strictly increasing* on $[a,b]$ if for any x_1 and x_2 from $[a,b]$ where $x_1 < x_2$, we have that $f(x_1) < f(x_2)$. Prove that if f is strictly increasing on $[a,b]$, then f is integrable.

Exercise 8.17. Consider Thomae's function h restricted to $[0,2]$. That is, $h:[0,2]\to\mathbb{R}$ is given by

$$h(x)=\begin{cases}1/n & \text{if } x\neq 0 \text{ and } x=m/n\in\mathbb{Q} \text{ in lowest terms with } n>0\\ 0 & \text{if } x\notin\mathbb{Q}\\ 1 & \text{if } x=0\end{cases}$$

This function is pictured on the cover of this book. In Example 6.7 we discussed how h is continuous at every irrational number and discontinuous at every rational number. In this question you will prove that Thomae's function is integrable. To do this, complete the following steps:

(a) Prove that $L(h,P)=0$ for any partition P of $[0,2]$.

(b) Note why h is bounded.

(c) Let $\varepsilon>0$. Determine whether there are finitely many points x such that $h(x)>\varepsilon/4$, or infinitely many such points. Explain your answer.

(d) Explain how to construct a partition P_ε of $[0,2]$ where $U(h,P_\varepsilon)<\varepsilon$. And prove that partition works.

(e) State which theorem completes this proof.

Exercise 8.18.

(a) Suppose f is continuous on $[a,b]$ and $\int_a^x f(t)dt=0$ for all $x\in[a,b]$. Prove that $f(x)=0$ for all $x\in[a,b]$.

(b) Suppose f is continuous on $[a,b]$ and $\int_a^x f(t)dt=\int_x^b f(t)dt$ for all $x\in[a,b]$. Prove that $f(x)=0$ for all $x\in[a,b]$.

Exercise 8.19. For each function f_k below, find a formula for $F_k(x)=\int_{-1}^x f_k(t)\,dt$. Where is F_k continuous? Where is F_k differentiable? Where does $F_k'=f_k$? You do not need to prove your answers.

(a) $f_1:[-1,1]\to\mathbb{R}$ given by $f_1(x)=|x|$.

(b) $f_2:[-1,1]\to\mathbb{R}$ given by

$$f_2(x)=\begin{cases}1 & \text{if } x<0\\ 2 & \text{if } x\geq 0.\end{cases}$$

(c) $f_3:[-1,1]\to\mathbb{R}$ given by

$$f_3(x)=\begin{cases}1 & \text{if } x\neq 0\\ 2 & \text{if } x=0.\end{cases}$$

Exercise 8.20. Suppose f and g are continuous functions on $[a,b]$, and $g(x)\geq 0$ for all $x\in[a,b]$. Prove that there exists some $c\in[a,b]$ such that

$$\int_a^b f(x)g(x)\,dx=f(c)\int_a^b g(x)\,dx.$$

Exercise 8.21. Suppose that f is continuous on $[0,1]$. Prove that

$$\lim_{n\to\infty}\int_0^1 f(x^n)\,dx=f(0).$$

Exercise 8.22. The *average value* of an integrable function f on an interval $[a,b]$ is defined to be

$$\text{avg}(f)=\frac{1}{b-a}\int_a^b f.$$

(a) Explain in your own words why this is a sensible definition of "average value."

(b) Prove that if f is the derivative of another function F, then avg(f) is the average rate of change of F over $[a,b]$.

(c) Prove that

$$\int_a^b \text{avg}(f)=\int_a^b f.$$

(d) Suppose that $m\leq f(x)\leq M$ for all $x\in[a,b]$. Prove that $m\leq \text{avg}(f)\leq M$.

(e) Prove that if f is continuous, then f achieves its average value: precisely, there is some $c\in[a,b]$ such that $f(c)=\text{avg}(f)$.

(f) Give an example to show that the previous part cannot be extended to all discontinuous f.

Exercise 8.23. Use Theorem 8.24 to provide a second proof (easier than the way outlined in Exercise 8.17) that Thomae's function is integrable.

Exercise 8.24. Assume that $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are integrable and $k\in\mathbb{R}$. In this exercise we will use Theorem 8.24 to prove that kf, $f+g$ and fg are also integrable. To that end, let $\mathcal{D}_f$ be the set of discontinuities of f; likewise for $\mathcal{D}_g$, $\mathcal{D}_{kf}$, $\mathcal{D}_{f+g}$ and $\mathcal{D}_{fg}$.

(a) Prove that if sets A and B have measure zero and $C\subseteq A\cup B$, then C has measure zero.

(b) Prove that $\mathcal{D}_{kf}\subseteq\mathcal{D}_f$, $\mathcal{D}_{f+g}\subseteq\mathcal{D}_f\cup\mathcal{D}_g$, and $\mathcal{D}_{fg}\subseteq\mathcal{D}_f\cup\mathcal{D}_g$.

(c) Explain why this implies that kf, $f+g$ and fg are all integrable.

Exercise 8.25. Give an example of each of the following, or state that no such example exists.

(a) A non-empty compact set of measure zero.

(b) A non-empty open set of measure zero.

Exercise 8.26.

(a) Assume f is integrable on $[a, b]$, and $f(x) \geq 0$ for all x. Moreover, assume that $\int_a^b f(x)\,dx > 0$. Prove that there are infinitely many points x for which $f(x) > 0$.

(b) Assume g is integrable on $[a, b]$ and $g(x) \geq 0$ for all x. If $g(x) > 0$ for an infinite number of values of x, must it be the case that $\int_a^b g(x)\,dx > 0$?

Exercise 8.27. Use the fundamental theorem of calculus (Theorem 8.32) and Darboux's theorem (Theorem 7.20) to give a second proof of the intermediate value theorem (Theorem 6.38).

Exercise 8.28.

(a) Suppose f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x. Prove that $\displaystyle\int_a^b f(x)\,dx \geq 0$.

(b) Suppose f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x. Prove that $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

(c) Suppose f is continuous on $[a, b]$ and $f(x) \geq 0$ for all x. Prove that if $f(x_0) > 0$ for some $x_0 \in [a, b]$, then $\displaystyle\int_a^b f(x)\,dx > 0$.

(d) Give an example of a function f on $[a, b]$ where $f(x) \geq 0$ for all x, $f(x_0) > 0$ for some $x_0 \in [a, b]$, and $\displaystyle\int_a^b f(x)\,dx = 0$.

Exercise 8.29.

(a) Let $F(x) = \displaystyle\int_0^x x f(t)\,dt$. What is $F'(x)$?

(b) Prove that if f is continuous, then

$$\int_0^x f(t)(x-t)\,dt = \int_0^x \left(\int_0^t f(s)\,ds\right)\,dt.$$

Exercise 8.30. Find a function f such that

$$\int_0^x t g(t)\,dt = x^2 + 2x^3.$$

Exercise 8.31. Prove that

$$\int_1^a \frac{1}{x}\,dx + \int_1^b \frac{1}{x}\,dx = \int_1^{ab} \frac{1}{x}\,dx.$$

Exercise 8.32. Prove that

$$\int_{ca}^{cb} f(x)\,dx = c \cdot \int_a^b f(cx)\,dx.$$

Exercise 8.33. Define a function $L\colon (0, \infty) \to \mathbb{R}$ by

$$L(x) = \int_1^x \frac{1}{t}\,dt.$$

You should intuitively think of this as the natural log function—although we haven't formally defined that yet.

(a) What is $L(1)$? Explain why L is differentiable and compute its derivative.

(b) Prove the identity $L(xy) = L(x) + L(y)$.

(c) Show that $L(x/y) = L(x) - L(y)$.

(d) Define

$$\gamma_n = \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n}\right) - L(n).$$

Prove that the sequence $(\gamma_n)_{n\geq 1}$ converges. The constant $e = \lim \gamma_n$ is called the *Euler–Mascheroni constant.*

(e) By considering the sequence $\gamma_{2n} - \gamma_n$ or otherwise, show that $L(2)$ is the limit of the alternating harmonic series.

(f) Prove that $L(x^y) = yL(x)$ for all $x \in (0, \infty)$, $y \in \mathbb{R}$. Deduce that $L(e^x) = x$ for all $x \in \mathbb{R}$. Deduce also that $\frac{d}{dx}e^x = e^x$.

Exercise 8.34. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Suppose further that f is continuous at every point in $[a, b]$ with the exception of a single point $x_0 \in (a, b)$. Prove that f is integrable on $[a, b]$.

Exercise 8.35. Suppose $f : [a,b] \to \mathbb{R}$ is integrable.

(a) Prove that there is some $x_0 \in [a,b]$ such that

$$\int_a^{x_0} f(t)\,dt = \int_{x_0}^b f(t)\,dt.$$

(b) Give an example showing that it's not always the case that $x_0 \in (a,b)$.

Exercise 8.36. Suppose $f : [a,b] \to \mathbb{R}$ is integrable. Follow the following steps to prove that f is continuous at infinitely many points (a dense set, in fact). Parts (a)–(e) will prove that there exists at least one point $c \in (a,b)$ where f is continuous at c.

(a) By a theorem from this chapter, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ with $U(f,P) - L(f,P) < b - a$. Which theorem gives this?

(b) Under the partition P, prove that $M_i - m_i < 1$ for some i. Let $I_1 = [x_{i-1}, x_i]$; that is,
$$\sup(\{f(x) : x \in I_1\}) - \inf(\{f(x) : x \in I_1\}) < 1.$$

(c) Prove that there exists a subinterval $I_2 \subseteq I_1$ where

$$\sup(\{f(x) : x \in I_2\}) - \inf(\{f(x) : x \in I_2\}) < \frac{1}{2}.$$

(d) Prove that, in general, there exists a subinterval $I_{k+1} \subseteq I_k$ where

$$\sup(\{f(x) : x \in I_{k+1}\}) - \inf(\{f(x) : x \in I_{k+1}\}) < \frac{1}{k+1}.$$

(e) Apply the *nested intervals theorem* (Exercise 1.34) to prove that there exists a point c at which f is continuous.

(f) Using what you just proved, show why f is continuous on a dense subset of $[a,b]$.

Exercise 8.37. Suppose that $f : [a,b] \to [0,\infty)$ is a continuous function with maximum value 2. Prove that

$$\lim_{n\to\infty} \left(\int_a^b f^n\right)^{1/n} = 2.$$

Exercise 8.38. Suppose that $f : [0,1] \to \mathbb{R}$ is integrable. Prove that there is a point $c \in (0,1)$ at which f is continuous.

Exercise 8.39.

(a) Prove that if f is integrable on $[a,b]$ and $m \le f(x) \le M$ for all x in $[a,b]$, then

$$\int_a^b f(x)\,dx = (b-a)\cdot\gamma$$

for some γ with $m \le \gamma \le M$.

(b) Prove that if f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = (b-a)\cdot f(x_0)$$

for some $x_0 \in [a,b]$.

(c) Show by example that (b) need not hold if f is not continuous.

(d) Prove the following more general result (part (c) is the special case $g(x) = 1$ for all x), known as the *general mean value theorem for integrals*: If f is continuous on $[a,b]$ and g is integrable and nonnegative on $[a,b]$, then

$$\int_a^b f(x)g(x)\,dx = f(x_0)\cdot\int_a^b g(x)\,dx$$

for some $x_0 \in [a,b]$.

Exercise 8.40. Say that a set $X \subseteq [a,b]$ has *content zero* if for every $\varepsilon > 0$ there are finitely many intervals $[a_1,b_1]$, $[a_2,b_2]$, ..., $[a_n,b_n]$ such that $X \subseteq \bigcup_{k=1}^n [a_k,b_k]$ and $\sum_{k=1}^n (b_k - a_k) < \varepsilon$. (The first condition says that the intervals cover X; the second says that their total length is small.)

(a) Show that every finite set has content zero.

(b) Show that the Cantor set $\mathcal{C}$ has content zero. (A definition of the Cantor set is in Example B.1 of Appendix B.

(c) Suppose that f is a bounded function on $[a,b]$ and that the set of points $x \in [a,b]$ at which f is discontinuous has content zero. Prove that f is integrable.

(d) Prove that the function

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathcal{C} \\ 0 & \text{if } x \notin \mathcal{C} \end{cases}$$

is integrable and compute $\int_0^1 h$. (Here $\mathcal{C}$ is the Cantor set.)

Exercise 8.41. Prove that the set $\{\int_1^b \frac{1}{x^2}\,dx : b \in [1,\infty)\}$ is bounded above and compute its supremum, which we might sensibly denote $\int_1^\infty \frac{1}{x^2}\,dx$.

Exercise 8.42. A function $s : [a,b] \to \mathbb{R}$ is called a *step function* if there is a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that s is constant on each interval (x_{i-1}, x_i).

(a) Prove that if f is integrable on $[a,b]$, then for any $\varepsilon > 0$ there is a pair of step functions s_1 and s_2 where $s_1(x) \le f(x) \le s_2(x)$ for all x, and

$$\int_a^b [f(x) - s_1(x)]\,dx < \varepsilon \qquad \text{and} \qquad \int_a^b [s_2(x) - f(x)]\,dx < \varepsilon.$$

(b) Suppose that for any $\varepsilon > 0$ there is a pair of step functions s_1 and s_2 where $s_1(x) \le f(x) \le s_2(x)$ for all x, and

$$\int_a^b [f(x) - s_1(x)]\,dx < \varepsilon \qquad \text{and} \qquad \int_a^b [s_2(x) - f(x)]\,dx < \varepsilon.$$

Prove that f is integrable.

(c) Give an example of a function f which is not a step function but for which $\int_a^b f(x)\,dx = L(f,P)$ for some partition P of $[a,b]$. You don't need to prove your answer.

Exercise 8.43. Prove that if $f : [a,b] \to \mathbb{R}$ is integrable, then for any $\varepsilon > 0$ there is a continuous function g where $g(x) \le f(x)$ for all x, and

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx < \varepsilon.$$

Exercise 8.44. Prove the *Cauchy-Bunyakovsky-Schqarz inequality*, which says that if f and g are integrable functions on $[a,b]$, then

$$\left(\int_a^b fg\right)^2 \le \left(\int_a^b f\right)\left(\int_a^b g\right).$$

— Open Questions —

Question 1. Under which conditions on f does the inequality

$$\int_a^b f^{\alpha+\beta}(x)\,dx \ge \left(\int_a^b (x-a)^\alpha f^\beta(x)\,dx\right)^\lambda$$

hold for some α, β and λ, and for such an f, what are the most general conditions on α, β and λ for which the above inequality holds?

Question 2. Under which conditions on f does the inequality

$$\frac{\int_a^b f^{\alpha+\beta}(x)\,dx}{\int_a^b f^{\alpha+\gamma}(x)\,dx} \ge \frac{\left(\int_a^b (x-a)^\alpha f^\beta(x)\,dx\right)^\delta}{\left(\int_a^b (x-a)^\alpha f^\gamma(x)\,dx\right)^\lambda}$$

hold for some $\alpha, \beta, \gamma, \delta$ and λ, and for such an f, what are the most general conditions on α, β, γ and λ for which the above inequality holds?

Question 3. Let $h : [0,\infty) \to \infty$ be a non-negative function and let $\alpha > 0$. Prove that if

$$\int_0^1 \frac{h(tx)}{x}(1-x)^{n-1}\,dx \le t^\alpha$$

for all $t \in [0,\infty)$, then

$$\int_0^\infty \frac{h(t)}{t + t^{2\alpha+1}}\,dt \le \frac{\pi}{2}\prod_{k=1}^{n-1}\left(1 + \frac{\alpha}{k}\right).$$