MAT 127B HW 1 Solutions

Wencin Poh
April 3, 2020

**Problem 1 (Exercise 5.2.3.)** By imitating the Dirichlet constructions in Section 4.1, construct a function on $\mathbb{R}$ that is differentiable at a single point.

**Solution** There are many candidates based on Dirichlet’s construction. One of the them is the function

$$g(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We first show that $g$ is differentiable at $x = 0$. By construction of $g(x)$, we observe that $0 \leq |g(x)| \leq |x|^2$ for all $x \in \mathbb{R}$, so that $0 \leq \frac{|g(x) - g(0)|}{x} = \frac{|g(x)|}{x} \leq |x|$. Furthermore, $\lim_{x \to 0} |x| = 0 = \lim_{x \to 0} 0$. By Squeeze Theorem, this implies that $\lim_{x \to 0} \frac{|g(x) - g(0)|}{x} = 0$.

Note that one cannot just consider left hand and right hand limits of the divided difference since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$ (there is no “universal” expression of values of $g(x)$ on either side).

We will then show that $g$ is not differentiable elsewhere, by showing that $g$ is not continuous elsewhere. This follows from either of the following proofs.

**Proof (High-level proof)** Note that $g(x) = x^2 1_\mathbb{Q}(x)$, where

$$1_\mathbb{Q}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

is Dirichlet’s function. It is known (from say, MAT 127A) that Dirichlet’s function is not continuous for all $x \neq 0$. As quotient of continuous functions are continuous (as long as one avoids division by 0), for all $x \neq 0$, $g$ cannot be continuous at $x$, since this implies that the Dirichlet’s function $1_\mathbb{Q}(x) = \frac{g(x)}{x^2}$ is also continuous at $x$. \[\Box\]

**Proof (Direct $\varepsilon-\delta$ proof)** To show that $g$ is not continuous at $x = a$ for all $a \neq 0$, we need to prove the following statement:

“There is an $\varepsilon > 0$ (possibly depending on $a$) such that for every $\delta > 0$, there is an $x \in \mathbb{R}$ with both $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$.”

Either $a$ is rational or irrational.

If $a \in \mathbb{Q}$, set $\varepsilon = \frac{|a|^2}{2}$. Since $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$, for every $\delta > 0$, there is an $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $x \in (a - \delta, a + \delta)$, that is, $|x - a| < \delta$.

Therefore, we have

$$|f(x) - f(a)| = |0 - a^2| = |a|^2 \geq \frac{|a|^2}{2} = \varepsilon.$$ 

On the other hand, if $a \notin \mathbb{Q}$, set $\varepsilon = \frac{|a|^2}{4}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every $\delta > 0$, there is an $x \in \mathbb{Q}$ such that $x \in (a - \delta', a + \delta')$, where $\delta' = \min\left(\delta, \frac{|a|}{2}\right)$. In other words, $|x - a| < \delta$ and $|x - a| < \frac{|a|}{2}$ both hold. By the reverse triangle inequality, the latter implies that $|x| \geq |a| - |a - x| = |a| - |x - a| > \frac{|a|}{2}$ or that $|x|^2 > \frac{|a|^2}{4}$.

Therefore, we have

$$|f(x) - f(a)| = |x^2 - 0| = |x|^2 > \frac{|a|^2}{2} = \varepsilon.$$ 

In either case, we have proved the required statement, that is, for all $a \neq 0$, $g$ is not continuous at $x = a$. \[\Box\]
In summary, we have constructed a function \( g \) that is differentiable at a single point (and nowhere else).

**Problem 2 (Exercise 5.2.4.)** Let 

\[
f_a(x) = \begin{cases} 
  x^a, & \text{if } x \geq 0, \\
  0, & \text{if } x < 0.
\end{cases}
\]

a. For which values of \( a \) is \( f \) continuous at zero?

b. For which values of \( a \) is \( f \) differentiable at zero? In this case, is the derivative function continuous?

c. For which values of \( a \) is \( f \) twice-differentiable?

**Solution**

a. Firstly, \( f_a(0) \) is defined for all values of \( a \) except when \( a = 0 \) (0\(^0\) is taken to be undefined). We have \( \lim_{x \to 0^-} f_a(x) = \lim_{x \to 0^+} 0 = 0 \), whereas \( \lim_{x \to 0^+} f_a(x) \) equals to

\[
\lim_{x \to 0^+} x^a = \begin{cases} 
  0, & \text{if } a > 0, \\
  1, & \text{if } a = 0, \\
  +\infty, & \text{if } a < 0.
\end{cases}
\tag{1}
\]

Therefore, we see that for \( \lim_{x \to 0} f_a(x) \) to exist, necessarily \( a > 0 \). Furthermore, whenever \( a > 0 \), \( \lim_{x \to 0} f_a(x) = f_a(0) = 0 \), so \( f_a \) is continuous at zero if and only if \( a > 0 \).

b. We use the definition of derivative at \( x = 0 \) and consider the one-sided limits \( \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} \) and \( \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} \). The latter is 0 since \( f(x) = 0 \) for all \( x < 0 \).

Moreover, we obtain

\[
\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^a - 0}{h} = \lim_{h \to 0^+} h^{a-1},
\]

hence, by comparing against Equation (1) (with \( a - 1 \) substituted for \( a \)), we argue similarly as in part (a) that \( f_a \) is differentiable at zero if and only if \( \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} \) exists (and equals 0), which holds if and only if \( a > 1 \).

In the case when \( a > 1 \), we claim that the derivative function is continuous. To see this, we first compute \( f_a'(x) \) for all \( x \in \mathbb{R} \). For all \( x > 0 \), we compute \( f_a'(x) = (x^a)' = ax^{a-1} \). For all \( x < 0 \), we compute \( f_a'(x) = 0' = 0 \).

Therefore, one has

\[
f_a'(x) = \begin{cases} 
  ax^{a-1}, & \text{if } x > 0, \\
  0, & \text{if } x \leq 0.
\end{cases}
\tag{2}
\]

Notice that \( f_a'(x) = 0 \) for all \( x \leq 0 \) (including 0). The functions \( ax^{a-1} \) for all \( x > 0 \) and 0 for all \( x < 0 \) are continuous. Additionally, we observe that \( \lim_{x \to 0^+} f_a(x) = 0 = f_a(0) \), which proves that \( f_a'(x) \) is continuous throughout \( \mathbb{R} \).

c. Based on Equation (2), we may compute the derivative function of \( f_a'(x) \), denoted \( f_a''(x) \) as follows. For all \( x > 0 \), we compute \( f_a''(x) = (ax^{a-1})' = a(a-1)x^{a-2} \). For all \( x < 0 \), we compute \( f_a'(x) = 0' = 0 \).

Moreover, for \( x = 0 \), we obtain

\[
\lim_{x \to 0^+} f_a''(x) = \lim_{h \to 0^+} \frac{f'(h) - f'(0)}{h} = \lim_{h \to 0^+} \frac{ah^{a-1} - 0}{h} = \lim_{h \to 0^+} ah^{a-2},
\]

and \( \lim_{x \to 0^-} f_a''(x) = 0 = f_a(0) \).

If we want \( f_a \) to be twice differentiable, then we need \( \lim_{h \to 0^+} ah^{a-2} = \lim_{x \to 0^+} f_a''(x) = \lim_{x \to 0^-} f_a''(x) = 0 \).

Comparing against Equation (1) (with \( a - 2 \) substituted for \( a \)), we see that this equality holds if and only if \( a > 2 \). This shows that \( f_a \) is twice differentiable if and only if \( a > 2 \).