b. Complete the proof of Darboux’s theorem stated earlier.

Solution  

b. We need to show the existence of \( c \in (a, b) \) with \( g'(c) = 0 \). Note first that \( g \) is differentiable on \([a, b]\), so \( g \) is continuous on \([a, b]\). Since \([a, b]\) is a compact set, \( g \) must attain a minimum in \([a, b]\).

By part (a), as \( g'(a) < 0 < g'(b) \), there exists points \( x, y \in (a, b) \) such that \( g(a) > g(x) \) and \( g(y) < g(b) \). Therefore, the minimum of \( g \) more specifically must exist within \((a, b)\) (as it cannot happen at \( x = a \) or \( x = b \)).

By the Interior Extremum Theorem (Theorem 5.2.6 in Abbott), since the minimum is attained at \( x_0 \in (a, b) \),

\[ g'(x_0) = 0. \]

Therefore, \( c \) is given by \( c = x_0 \), completing the proof.

Problem 2 (Exercise 5.2.8.)  
Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

a. If a derivative function is not constant, then the derivative must take on some irrational value.

Solution  

a. This conjecture is false in general, as the domain of a differentiable function \( f \) is allowed to be not connected. One can define \( f : (0, 1) \cup (1, 2) \to \mathbb{R} \) as

\[ f(x) = \begin{cases} 
  x, & \text{if } 0 < x < 1, \\
  2x, & \text{if } 1 < x < 2.
\end{cases} \]

and compute the derivative as

\[ f'(x) = \begin{cases} 
  1, & \text{if } 0 < x < 1, \\
  2, & \text{if } 1 < x < 2.
\end{cases} \]

In particular, there are no values of \( x \) in the domain of \( f \) with \( f'(x) \notin \mathbb{Q} \).

The text in this font is only for explanation purpose and is not expected to appear in a solution.

On the other hand, if the domain of \( f \) is a connected interval, then this conjecture is true by Darboux’s theorem. If \( x_1, x_2 \) are distinct values in the domain of \( f \) with \( x_1 < x_2 \), \( f'(x_1) \neq f'(x_2) \) and \( f'(x_1) \neq f'(x_2) \), without loss of generality we may assume that \( f'(x_1) < f'(x_2) \), then for every irrational value \( \alpha \) in the interval \((f'(x_1), f'(x_2))\), we may apply Darboux’s theorem to find a \( c \in (x_1, x_2) \) such that \( f'(c) = \alpha \).

b. This conjecture is false. Consider the function

\[ f(x) = \begin{cases} 
  x + 10x^2 \sin(1/x), & \text{if } x \neq 0, \\
  0, & \text{if } x = 0,
\end{cases} \]

defined on \((-1, 1)\).
$f$ is differentiable on $(-1, 1)$ as we may compute the derivative function as

$$f'(x) = \begin{cases} 1 + 20x \sin(1/x) - 10 \cos(1/x), & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Note that if $c = 0$, $f'(c) = 1 > 0$. For every $\delta > 0$, by the Archimedean property of real numbers, we may choose a positive integer $N$ such that $\frac{1}{2\pi N} < \delta$. Hence, if we set $x = \frac{1}{2\pi N}$, we have $x \in (-\delta, \delta)$ but $f'(x) = 1 + 20x \sin(2\pi N) - 10 \cos(2\pi N) = 1 + 20(0) - 10(1) = -9 < 0$.

In particular, there does not exist a $\delta$-neighborhood $V_\delta(c) = (-\delta, \delta)$ around $c$ with $f'(x) > 0$ for all $x \in V_\delta(c)$.

c. This conjecture is true. We shall prove that for every $\varepsilon > 0$, $|L - f'(0)| < \varepsilon$, so that in fact, $L = f'(0)$.

Let $\varepsilon > 0$ be arbitrary. Since $f$ is differentiable at $0$, there is a $\delta_1 > 0$ such that for all $x$ in the domain of $f$, $0 < |x - 0| < \delta_1$ implies that $\left| \frac{f(x) - f(0)}{x - 0} - f'(0) \right| < \frac{\varepsilon}{2}$. Since $\lim_{x \to 0} f'(x) = L$, there is a $\delta_2 > 0$ such that for all $x$ in the domain of $f$, $0 < |x - 0| < \delta_2$ implies that $\left| \frac{f(x) - f(0)}{x - 0} - L \right| < \frac{\varepsilon}{2}$.

Now, if we set $\delta = \min(\delta_1, \delta_2)$ and use Triangle Inequality, for all $x$ in the domain of $f$, $0 < |x - 0| < \delta$ implies that

$$|L - f'(0)| \leq \left| L - \frac{f(x) - f(0)}{x - 0} \right| + \left| \frac{f(x) - f(0)}{x - 0} - f'(0) \right|$$

$$= \left| \frac{f(x) - f(0)}{x - 0} - L \right| + \left| \frac{f(x) - f(0)}{x - 0} - f'(0) \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

Therefore, $|L - f'(0)| < \varepsilon$ for every $\varepsilon > 0$, so in fact, $|L - f'(0)| = 0$ or that $L = f'(0)$.

d. The conjecture is false as $f$ is allowed to be discontinuous at $x = 0$.

Consider the function $f : (-1, 1) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Observe that $f$ is differentiable on $(-1, 1)\setminus\{0\}$ at least, and $f'(x) = 0$ for all $x \neq 0$, so that $\lim_{x \to 0} f'(x) = 0$. However, $f$ is not continuous at $x = 0$ since $\lim_{x \to 0} f(x)$ does not exist, therefore, $f'(0)$ does not exist and does not equal 0.

On the other hand, if $f$ is continuous at $x = 0$, this conjecture is true under that setting. The proof for why it works require the use of the Mean Value Theorem.