Problem 1 (Exercise 6.2.10.) Let \( f \) be uniformly continuous on all of \( \mathbb{R} \), and define a sequence of functions by \( f_n(x) = f \left( x + \frac{1}{n} \right) \). Show that \( f_n \to f \) uniformly. Give an example to show that this proposition fails if \( f \) is only assumed to be continuous and not uniformly continuous on \( \mathbb{R} \).

Solution Text in italicized font is meant to be explanatory and is not expected to be part of a proof.

We need to show that the following statement holds:

\[
\forall \varepsilon > 0 \ \exists N \in \mathbb{Z}_{>0} \ \forall x \in \mathbb{R} \ \forall n \in \mathbb{Z}_{>0} : \ n > N \implies \left| f \left( x + \frac{1}{n} \right) - f(x) \right| < \varepsilon .
\]

In other words, “For all \( \varepsilon \), there is a positive integer \( N \), such that for all \( x \in \mathbb{R} \) and for all positive integers \( n \), \( n > N \) implies \( \left| f \left( x + \frac{1}{n} \right) - f(x) \right| < \varepsilon \).”

Let \( \varepsilon > 0 \) be arbitrary. Since \( f \) is uniformly continuous, there is a \( \delta > 0 \) such that for all \( a, b \in \mathbb{R} \), \( |a - b| < \delta \) implies that \( |f(a) - f(b)| < \varepsilon \).

Define \( N \) as the smallest positive integer such that \( \frac{1}{N} < \delta \). This is possible by the Archimedean property of the real numbers. For all \( x \in \mathbb{R} \) and for all integers \( n \), consider \( a = x + \frac{1}{n} \) and \( b = x \). Now, \( n > N \) implies that \( \frac{1}{n} < \frac{1}{N} \) so that \( |a - b| = \frac{1}{n} < \delta \).

Uniform continuity of \( f \) implies that \( \left| f \left( x + \frac{1}{n} \right) - f(x) \right| < \varepsilon \). It follows that \( f_n \to f \) uniformly.

Before we present a counterexample showing that the proposition fails if \( f \) is allowed to be continuous but not uniformly continuous, recall that we need to show that the following statement holds for such an \( f \):

\[
\exists \varepsilon > 0 \ \forall N \in \mathbb{Z}_{>0} \ \exists x \in \mathbb{R} \ \exists n \in \mathbb{Z}_{>0} : \ n > N \text{ and } \left| f \left( x + \frac{1}{n} \right) - f(x) \right| \geq \varepsilon .
\]

In other words, “There is an \( \varepsilon \) such that for all positive integers \( N \), there exists \( x \in \mathbb{R} \) and there is a positive integer \( n \) such that \( n > N \) but \( \left| f \left( x + \frac{1}{n} \right) - f(x) \right| < \varepsilon \).”

Now, consider \( f(x) = x^2 \) on \( \mathbb{R} \). Set \( \varepsilon = 1 \). For every positive integer \( N \), set \( x = \frac{N+1}{2} \) and \( n = N+1 \). It is clear that \( n > N \), but then

\[
\left| f \left( x + \frac{1}{n} \right) - f(x) \right| = \left| \left( x + \frac{1}{n} \right)^2 - x^2 \right|
= \left| \frac{2x}{n} + \frac{1}{n^2} \right|
= \frac{N+1}{N+1} + \frac{1}{(N+1)^2}
= 1 + \frac{1}{(N+1)^2}
\geq 1 .
\]

It follows that \( (f_n) \) does not converge uniformly to \( f \) on \( \mathbb{R} \).
Problem 2 (Exercise 6.2.11.) Assume \((f_n)\) and \((g_n)\) are uniformly convergent sequences of functions.

a. Show that \((f_n + g_n)\) is a uniformly convergent sequence of functions.

b. Give an example to show that the product \((f_ng_n)\) may not converge uniformly.

c. Prove that if there exists an \(M > 0\) such that \(|f_n| \leq M\) and \(|g_n| \leq M\) for all \(n \in \mathbb{N}\), then \((f_ng_n)\) does converge uniformly.

Solution

a. Suppose \(A\) is the domain of both \(f\) and \(g\). Let \(\varepsilon > 0\). Since \((f_n)\) converges uniformly to \(f\), there is a positive integer \(N_1\) such that for all \(x \in A\) and for all integers \(n, N_1\) implies that \(|f_n(x) - f(x)| < \frac{\varepsilon}{2}\).

Since \((g_n)\) converges uniformly to \(g\), there is a positive integer \(N_2\) such that for all \(x \in A\) and for all integers \(n, N_2\) implies that \(|g_n(x) - g(x)| < \frac{\varepsilon}{2}\).

Set \(N = \max(N_1, N_2)\). For all \(x \in A\) and for all integers \(n\), we have the following to hold if \(n > N\):

\[
|\(f_n(x) + g_n(x)\) - (f(x) + g(x))| = |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\
\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon.
\]

It follows that \((f_n + g_n)\) uniformly converges to \(f + g\) on \(A\).

b. For all integers \(n \geq 1\), let \((f_n)\) and \((g_n)\) be sequences of functions on \(\mathbb{R}\) given by \(f_n(x) = x^2\) and \(g_n(x) = \frac{1}{n}\).

We claim that both \((f_n)\) and \((g_n)\) are uniformly convergent on \(\mathbb{R}\), but then \((f_ng_n)\) is not uniformly convergent on \(\mathbb{R}\).

As \((f_n)\) is a constant sequence of functions \(x\), it automatically converges uniformly to \(f(x) = x\) on \(\mathbb{R}\).

Meanwhile, \((g_n)\) is a sequence of constant functions that converges pointwise to the zero function, \(g(x) = 0\). As the rate of convergence does not depend on \(x \in \mathbb{R}\), we conclude that \((g_n)\) converges to \(g\) on \(\mathbb{R}\).

Now, if \((f_ng_n)\) were to converge uniformly, it would have to converge to \(f(x)g(x) = 0\). However, we have \(|f_n(x)g_n(x) - f(x)g(x)| = \left|\frac{x}{n}\right|\), so if we set \(\varepsilon = 1\), then for all integers \(N > 0\), we may choose \(x = n = N + 1\) so that \(n > N\) but then

\[
|f_n(x)g_n(x) - f(x)g(x)| = \left|\frac{x}{n}\right| = \frac{N + 1}{N + 1} = 1 \geq \varepsilon.
\]

Thus, we conclude that \((f_ng_n)\) does not converge uniformly on \(\mathbb{R}\).

c. Suppose that \(A\) is the domain of both \(f\) and \(g\). We first prove that \(|g(x)| < M + 1\) for all \(x \in A\). Since \((g_n)\) uniformly converges to \(g\), there is an \(N'\) such that for all \(x \in A\) and for all integers \(n, N'\) implies that \(|g_n(x) - g(x)| < 1\). In particular, we have

\[
|g(x)| < |g_{2N}(x)| + |g(x) - g_{2N}(x)| < M + 1.
\]

Let \(\varepsilon > 0\) be arbitrary. Since \((f_n)\) converges uniformly to \(f\), there is a positive integer \(N_1\) such that for all \(x \in A\) and for all integers \(n, N_1\) implies that \(|f_n(x) - f(x)| < \frac{\varepsilon}{2M}\). Since \((g_n)\) converges uniformly to \(g\), there is a positive integer \(N_2\) such that for all \(x \in A\) and for all integers \(n, N_2\) implies that \(|g_n(x) - g(x)| < \frac{\varepsilon}{2(M + 1)}\).

Hence, we have

\[
|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\
\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\
< M \cdot \frac{\varepsilon}{2M} + (M + 1) \cdot \frac{\varepsilon}{2(M + 1)} \\
= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon.
\]

Thus, \((f_ng_n)\) uniformly converges to \(f(x)g(x)\) on \(A\).
Problem 3 (Exercise 6.2.12.) Theorem 6.2.6. has a partial converse. Assume \( f_n \to f \) pointwise on a compact set \( K \) and assume that for each \( x \in K \) the sequence \( f_n(x) \) is increasing. Follow these steps to show that if \( f_n \) and \( f \) are continuous on \( K \), then the convergence is uniform.

a. Note that as the sequence \( f \) are continuous on set \( K \).

b. Assume the hypothesis in part (a). As in the problem, let \( \varepsilon > 0 \) be arbitrary, and define \( K_n = \{ x \in K \mid g_n(x) \geq \varepsilon \} \). Argue that \( K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots \) is a nested sequence of compact sets, and use this observation to finish the argument.

Solution

a. Note that as the sequence \( f_n(x) \) is increasing for all \( x \in K \) and \( f(x) = \lim_{n \to \infty} f_n(x) \), one has \( f(x) \geq f_n(x) \) for all \( x \in K \). Furthermore, for all positive integers \( m, n \), we have \( m < n \) to imply that \( f(x) - f_n(x) \leq f(x) - f_m(x) \). Also, note that \( f - f_n \) is continuous as it is a difference of two continuous functions. Then, the hypothesis translates into the following: “Assume that \( g_n \to 0 \) pointwise on a compact set \( K \) and assume that for each \( x \in K \), the sequence \( g_n(x) \) is decreasing. Assume further that \( g_n \) is continuous for all positive integers \( n \).”

b. Assume the hypothesis in part (a). As in the problem, let \( \varepsilon > 0 \) be arbitrary, and define \( K_n = \{ x \in K \mid g_n(x) \geq \varepsilon \} \).

We claim that for all positive integers \( r \), \( K_r \supseteq K_{r+1} \). Indeed, if we let \( r \) be arbitrary, and suppose that \( x \in K_{r+1} \), then \( g_{r+1}(x) \geq \varepsilon \). However, since the sequence \( g_n(x) \) is decreasing \( g_r(x) \geq g_{r+1}(x) \geq \varepsilon \), so that \( x \in K_r \) too.

Moreover, each \( K_n \) is a preimage of \([\varepsilon, \infty)\) under the continuous function \( g_n \). Since \([\varepsilon, \infty)\) is closed and preimage of closed sets under a continuous function is closed, each \( K_n \) is a closed subset of the compact set \( K \). Note also that closed subsets of compact sets is again compact.

It follows that \( K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots \) is a nested sequence of compact sets. Therefore, if we consider \( U_n = K \setminus K_n \), each \( U_n \) is an open set and \( \{ U_n \mid n \in \mathbb{Z}_{>0} \} \) is a family of open sets that cover \( K \). By the compactness of \( K \), there is a finite subcover by say, the open set \( U_{i_1}, U_{i_2}, \ldots, U_{i_m} \) for some positive integer \( m \). Since \( U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots \) and that \( U_{i_1}, U_{i_2}, \ldots, U_{i_m} \) cover \( K \), we conclude that indeed \( U_{i_m} = K \) and hence also \( U_n = K \) for all \( n > i_m \). Note that \( U_n = \{ x \in K \mid g_n(x) < \varepsilon \} \) for every positive integer \( n \).

Therefore, we have proved that for every \( \varepsilon > 0 \), there is a positive integer \( N = i_m \) such that for all \( x \in K \) and for all integers \( n, n > N \) implies that \( g_n(x) < \varepsilon \). This completes the proof that \( g_n \to 0 \) uniformly, hence \( f_n \to f \) uniformly.