MAT 127B HW 16 Solutions

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**Problem 1** Show that \( \eta(x) \) in Example 10.33 is smooth but not real analytic.

**Solution** *Text in italicized font is meant to be explanatory and is not expected to be part of a proof.*

Recall that \( \eta \) is given by \( \eta(x) = \phi(1 - x^2) \), where

\[
\phi(x) = \begin{cases} 
  e^{-1/x}, & \text{if } x > 0, \\
  0, & \text{otherwise.}
\end{cases}
\]

Furthermore, we know that \( \phi \) is smooth and \( \phi^{(n)}(0) = 0 \) for all \( n \geq 0 \).

We first check that \( \eta \) is smooth by showing that \( \eta^{(n)}(x) \) is defined for all \( x \in \mathbb{R} \) and for all integers \( n \geq 1 \).

Indeed, it is relatively straightforward to verify, by repeatedly using the Chain Rule, that for every \( n \geq 1 \), there are polynomials \( f_1(x), f_2(x), \ldots, f_n(x) \) such that \( \eta^{(n)}(x) = \sum_{k=1}^{n} f_k(x)\phi^{(k)}(1 - x^2) \). To see this more carefully, we have \( \eta'(x) = (-2x)\phi'(1 - x^2) \) by using the Chain Rule. Letting \( m \) be arbitrary with \( m \geq 1 \) and assuming that \( \eta^{(m)}(x) = \sum_{k=1}^{m} f_k(x)\phi^{(k)}(1 - x^2) \) for some polynomials \( f_1(x), f_2(x), \ldots, f_n(x) \), we use the Chain Rule again to obtain

\[
\eta^{(m+1)}(x) = \sum_{k=1}^{m} f_k'(x)\phi^{(k)}(1 - x^2) - 2xf_k(x)\phi^{(k+1)}(1 - x^2)
\]

\[
= f_1'(x)\phi'(1 - x^2) + \sum_{k=2}^{m} (f_k'(x) - 2xf_{k-1}(x))\phi^{(k)}(1 - x^2) - 2xf_m(x)\phi^{(m+1)}(1 - x^2)
\]

\[
= \sum_{k=1}^{m+1} g_k(x)\phi^{(k)}(1 - x^2),
\]

where \( g_1(x) = f_1'(x) \), \( g_k(x) = f_k'(x) - 2xf_{k-1}(x) \) for \( k = 2, 3, \ldots, m \) and \( g_{m+1}(x) = -2xf_m(x) \) are polynomials.

This shows that \( \eta \) is smooth and in fact, \( \eta^n(1) = \sum_{k=1}^{n} f_k(1)\phi^{(k)}(0) = 0 = \sum_{k=1}^{n} f_k(-1)\phi^{(k)}(0) = \eta^{(n)}(-1) \) for all integers \( n \geq 0 \).

However, the power series associated to \( \eta \) centered at 1 is

\[
\sum_{n=0}^{\infty} \frac{\eta^{(n)}(1)}{n!} (x-1)^n = 0,
\]

yet \( \eta(x) = e^{-1/(1-x^2)} \neq 0 \) for all \( |x| < 1 \). Thus, there is no \( \varepsilon \)-neighborhood of 1 where \( \eta(x) = \sum_{n=0}^{\infty} \frac{\eta^{(n)}(1)}{n!} (x-1)^n \) for all \( x \) in the neighborhood, so \( \eta \) is smooth, but not analytic. *A similar argument can be made if one uses -1 instead of 1.*
**Problem 2** Find a power series \( F(x) = \sum_{n=1}^{\infty} a_n x^n \) with \( F(0) = 1 \) and \( F(x) \cdot F(x) = F'(x) \).

**Solution** Let \( F(x) \) be such a power series. We claim that for all \( n \geq 0 \), \( F^{(n)}(x) = n! [F(x)]^{n+1} \).

Let \( P(n) \) be the statement that \( F^{(n)}(x) = n! [F(x)]^{n+1} \) for each integer \( n \geq 0 \). \( P(0) \) holds because we have the convention that \( F^{(0)}(x) = F(x) \).

Let \( k \geq 0 \) be an arbitrary integer and assume that \( F^{(k)}(x) = k! [F(x)]^{k+1} \). Then, by Chain Rule, we have

\[
F^{(k+1)}(x) = k!(k+1)[F(x)]^{k} [F'(x)] = (k+1)! [F(x)]^{k} [F(x)]^2 = (k+1)! [F(x)]^{k+2}.
\]

By induction, we have proved that \( F^{(n)}(x) = n! [F(x)]^{n+1} \) for all \( n \geq 0 \).

The power series associated to \( F(x) \) is then

\[
\sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n! [F(0)]^{n+1}}{n!} x^n = \sum_{n=0}^{\infty} x^n.
\]

It is well-known that the geometric series \( \sum_{n=0}^{\infty} x^n \) converges if and only if \( |x| < 1 \), so \( F(x) = \sum_{n=0}^{\infty} x^n \) is well-defined and is the desired power series.

\( \Box \)