Problem 1. 7.2.4

a) For the forward direction, suppose that \( f \) is integrable on \([a, b]\). Then by the Cauchy integrability condition, for all \( n \in \mathbb{N} \) there is a partition \( P_n \) such that \( U(f, P_n) - L(f, P_n) \leq \frac{1}{n} \). Then \( 0 \leq \lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] \leq \lim_{n \to \infty} \frac{1}{n} \), so \( \lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0 \) by the squeeze theorem.

For the reverse direction, let \( \epsilon > 0 \), and suppose we have a sequence of partitions \( P_n \) such that \( \lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0 \). Then by the definition of limit, there is some \( N \) such that \( U(f, P_N) - L(f, P_N) = |U(f, P_N) - L(f, P_N)| < \epsilon \). Therefore \( f \) satisfies the Cauchy integrability condition, so \( f \) is integrable.

b) We have
\[
U(f, P_n) = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{k}{n} = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{n(n+1)}{2n^2}.
\]
\[
L(f, P_n) = \sum_{k=0}^{n-1} \frac{1}{n} \cdot \frac{k}{n} = \frac{1}{n^2} \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2n^2}.
\]

c) We have
\[
\lim_{n \to \infty} U(f, P_n) - L(f, P_n) = \lim_{n \to \infty} \frac{1}{n} = 0,
\]
so \( f \) is integrable by part a).

Problem 2. 7.2.5

Let \( \epsilon > 0 \). Since \( f_n \to f \) uniformly, there exists an \( N \) such that when \( n \geq N \), \( |f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)} \). Since \( f_N \) is integrable, by the Cauchy integrability criterion there is a partition \( P \) such that \( U(f_N, P) - L(f_N, P) < \frac{\epsilon}{3} \). Since \( f \) and \( f_n \) differ by at most \( \frac{\epsilon}{3(b-a)} \) on all of \([a, b]\), it follows that \( |U(f, P) - U(f_N, P)| < \frac{\epsilon}{3} \) and \( |L(f, P) - L(f_N, P)| < \frac{\epsilon}{3} \).

Putting all of this together, we have
\[
U(f, P) - L(f, P) = U(f, P) - U(f_N, P) + U(f_N, P) - L(f_N, P) + L(f_N, P) - L(f, P)
\leq |U(f, P) - U(f_N, P)| + |U(f_N, P) - L(f_N, P)| + |L(f_N, P) - L(f, P)|
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}
= \epsilon,
\]
so \( f \) is integrable by the Cauchy integrability criterion.