Homework 19 Solutions

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Problem 1 (Exercise 7.3.2) Let $g(x)$ be the Dirichlet function. Construct a sequence $g_n(x)$ of Riemann integrable functions with $g_n \to g$ pointwise on $[0,1]$. This demonstrates that the pointwise limit of integrable functions need not be integrable. Compare this to exercise 7.2.5.

Solution
Since the rationals are countable, we may index them and write $\mathbb{Q} = \{q_1, q_2, \ldots \}$. Now, define $Q_n$ to be the set containing the first $n$ rationals and let

$$g_n(x) = \begin{cases} 1 & x \in Q_n \\ 0 & \text{otherwise} \end{cases}$$

Thus, $\forall x \in [0,1], \exists N$ such that if $n > N$, then $g_n(x) = g(x)$. So, $g_n \to g$ pointwise on $[0,1]$. Also, for each $n$, $g_n$ has only finite discontinuities, so by Theorem 11.53, it is integrable.

In exercise 7.2.5, we showed that uniform convergence preserves integrability. Keep in mind, both of these problems only show the results on compact domains.

Problem 2 (Exercise 7.3.3) An alternate explanation for why a function $f$ on $[a, b]$ with a finite number of discontinuities is integrable. Supply the missing details.

Embed each discontinuity in a sufficiently small open interval and let $O$ by the union of these intervals. Explain why $f$ is uniformly continuous on $[a, b] \setminus O$, and use this to finish the argument.

Solution
Note that this problem can be solved much like 7.3.4(a). That is, we know that $[a, b] \setminus O$ is compact, and continuous functions on compact sets are integrable by Thm 7.2.9. So, we compare the maximum difference between $\int_{[a,b] \setminus O} f(x)$ and a carefully defined Riemann sum over all of $[a, b]$. The difference can be expressed as the areas of rectangles whose heights are the maximum size of discontinuity in $f$ and whose widths are arbitrarily small. Thus, the difference can be made arbitrarily small and the Riemann sum behaves well enough to satisfy the definition of integrability.

Since there’s a more formal version of that argument in the next problem, and Theorem 11.61 was covered in lecture, let’s use that theorem to solve this problem:

Theorem 11.61 states that a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and the set of points at which it is discontinuous has Lebesgue measure zero.

Say there are $n$ discontinuities in $f$, and let the set of discontinuities be $D = \{x_1, x_2, \ldots, x_n\}$.

Assuming $f$ is bounded (not explicitly given, but Abbot has not dealt with any unbounded functions yet), we just need to show that $D$ has Lebesgue measure zero. By the definition given in lecture, this means that...
Problem 3 (Exercise 7.3.4) Assume \( f : [a, b] \to \mathbb{R} \) is integrable.

(a) Show that if one value of \( f(x) \) is changed, \( f \) is still integrable and integrates to the same value as before.

(b) Show that the observation in (a) holds if a finite number of values of \( f \) are changed.

(c) Find an example to show that by altering a countable number of values, \( f \) may fail to be integrable.

Solution

(a) We know from Thm 11.23 that \( \forall \epsilon, \) there is a partition \( P_\epsilon \) such that \( U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon / 3. \)

Let’s define a function with one value changed from \( f \). Let \( g(x) = \begin{cases} f(x) & x \neq c \\ z & x = c \end{cases} \)

We wish to show that \( g \) is integrable, so we’ll need to quantify the difference in the upper and lower Riemann sums. To that end, let \( k = |f(c) - z|. \)

Now, given some \( \epsilon \) and the associated \( P_\epsilon \) from above, define the refinement \( P_{\epsilon, r} = P_\epsilon \cup \{c - \epsilon/6k, c + \epsilon/6k\}. \)

It follows that \( U(g, P_{\epsilon, r}) \leq U(f, P_\epsilon) + 2k \frac{\epsilon}{6k} = U(f, P_\epsilon) + \frac{\epsilon}{3}. \)

It helps to draw \( f \) and \( g \) to see this. Intuitively, in moving from \( f \) to \( g \) and refining the partition in this way, the most we can possibly add to our Riemann sum is 2 rectangles of height \( k \) and width \( \epsilon/6k. \)

Similarly, \( L(g, P_{\epsilon, r}) \geq L(f, P_\epsilon) - \frac{\epsilon}{3}. \)

Putting all this together, we have

\[
U(g, P_{\epsilon, r}) - L(g, P_{\epsilon, r}) \leq (U(f, P_\epsilon) - L(f, P_\epsilon)) + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon
\]

So, we’ve met the Cauchy criterion, and shown that \( g \) is integrable. It remains to show that \( f \) and \( g \) integrate to the same value.

Let \( \int_a^b f(x) = I. \) By definition, we have \( I = \inf U(f, P), \) where the infimum is taken over all partitions of \([a, b].\) From the previous argument, we have that \( U(g, P_{\epsilon, r}) \leq U(f, P_\epsilon) + \frac{\epsilon}{3}. \) Since this holds for all \( \epsilon, \) it follows that \( \inf U(g, P) = I. \) Thus, \( \int_a^b g(x) = I, \) as desired.

(b) This can be shown with induction. The base case is done in part (a). So, we next let \( g \) be a function that equals \( f \) except at \( k \) points in \([a, b]\) and assume that \( g \) is integrable and \( \int_a^b f(x) = \int_a^b g(x). \) We wish to show this still holds if \( k + 1 \) values are changed.

Let \( h(x) = \begin{cases} g(x) & x \neq c \\ z & x = c \end{cases}, \) with \( z \neq f(c), \) so \( h \) is a function with \( k + 1 \) values changed from \( f. \) By part (a), since \( g \) is integrable, we know that \( h \) is integrable and \( \int_a^b h(x) = \int_a^b g(x) = \int_a^b f(x). \)

So, the inductive step is verified, and we have that the observations from part (a) hold for any finite number of changed values.
(c) See problem 1. The Dirichlet function is such an example. It can be thought of as the 0 function, with countably many values changed to 1.

A couple of interesting notes: By moving only countably many points, we created uncountable discontinuities (the Dirichlet function is discontinuous everywhere). However, this is not always the case. Consider Thomae’s function, which also has each of its “rational points” moved away from 0. It is continuous at each irrational, and thus has only countable discontinuities. The difference is that in creating the Dirichlet function, we changed each rational value by 1, but in creating Thomae’s function, most values were moved by an arbitrarily small amount.