Problem 1 (Exercise 7.3.5) Let
\[ f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \]
Show that \( f \) is integrable on \([0,1]\) and compute \( \int_0^1 f(x) \).

**Solution**
Let \( r \) be a number such that \( 0 < r < 1 \). On \([r,1]\), \( f \) is nonzero at only a finite number of points. Thus, from Prop 11.46, we know that \( f \) is integrable on \([r,1]\) and that \( \int_r^1 f = \int_r^1 0 = 0 \).

Then, by Prop 11.50, \( f \) is integrable on \([0,1]\) and \( \int_0^1 f = \lim_{r \to 0} \int_r^1 f = 0 \).

Problem 2 (Exercise 7.4.4) Decide which of the following conjectures is true and supply a short proof. For those that are not true, give a counterexample.

(a) If \( |f| \) is integrable on \([a,b]\) then \( f \) is also integrable on this set.

(b) Assume \( g \) is integrable and \( g \geq 0 \) on \([a,b]\). If \( g(x) > 0 \) for an infinite number of points \( x \in [a,b] \), then \( \int_a^b g > 0 \).

(c) If \( g \) is continuous on \([a,b]\), and \( g \geq 0 \) with \( g(x_0) > 0 \) for at least one point \( x_0 \in [a,b] \), then \( \int_a^b g > 0 \).

(d) If \( \int_a^b f > 0 \), there is an interval \([c,d] \subseteq [a,b] \) and \( \delta > 0 \) such that \( f(x) \geq \delta \) for all \( x \in [c,d] \).

**Solution**
(a) False. As a counterexample, consider something akin to the Dirichlet function: \( f = \begin{cases} -1 & x \in \mathbb{Q} \\ 1 & \text{otherwise.} \end{cases} \)

(b) False. Thomae’s function is a counterexample and details have been worked out in previous homeworks.

(c) True. If \( g(x_0) = m \), by continuity we know that there is a delta neighborhood \( V_\delta(x_0) \) such that \( g(x) > m/2 \) for all \( x \in V_\delta(x_0) \). Thus, \( \int_{x_0-\delta}^{x_0+\delta} g(x) > \delta m > 0 \). Since \( g \geq 0 \), it follows that \( \int_a^b g > 0 \).

Also note that this is the contrapositive of Prop 11.42, which is proven in Hunter.

(d) True. We’ll prove the contrapositive. So, assume that for all intervals \([c,d]\) and for any \( \delta > 0 \), there is an \( x \in [c,d] \) with \( f(x) < \delta \). In other words, all intervals contain an arbitrarily small (or negative) function value. Then, for any partition \( P \) of \([a,b]\), we have \( L(f,P) \leq 0 \), which implies that \( L(f) \leq 0 \). So, if \( f \) is integrable, \( \int_a^b f \leq 0 \).
Problem 3 (Exercise 7.6.3) Show that any countable set has measure 0.

Solution
Let $X = \{x_1, x_2, \ldots\}$ be an arbitrary countable subset of $\mathbb{R}$. Then define intervals

$$A_n = \left(x_n - \frac{\delta}{2n^2}, x_n + \frac{\delta}{2n^2}\right).$$

Clearly, $\cup A_n$ contains $X$. We also have

$$\sum_{n=1}^{\infty} |A_n| = \sum_{n=1}^{\infty} \frac{\delta}{n^2} = \frac{\delta \pi^2}{6}.$$

So, for any $\epsilon > 0$, we can take $\delta < \frac{6\epsilon}{\pi^2}$ to make the above sum less than $\epsilon$. Thus, $X$ has measure 0.