Problem 1 (Exercise 7.5.1) We have seen that not every derivative is continuous, but explain how we at least know that every continuous function is a derivative.

Solution Given a continuous function $f : [a, b] \to \mathbb{R}$, define the function $F : [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f(t) \, dt$.

Since $f$ is continuous on $[a, b]$, $f$ is also Riemann integrable on $[a, b]$. Furthermore, as $f$ is continuous at $c$ for all $c \in [a, b]$, by the Fundamental Theorem of Calculus (Theorem 7.5.1(ii) of Abbott or Theorem 12.6 of Hunter), $F$ is differentiable at $c$ for all $c \in [a, b]$, and in fact, $F'(c) = f(c)$ for all such $c$.

Problem 2 (Exercise 7.5.4) Let $H(x) = \int_1^x \frac{1}{t} \, dt$, where we consider only $x > 0$.

a. What is $H(1)$? Find $H'(x)$.

b. Show that $H$ is strictly increasing; that is, show that if $0 < x < y$, then $H(x) < H(y)$.

c. Show that $H(cx) = H(c) + H(x)$. (Think of $c$ as a constant and differentiate $g(x) = H(cx)$.)

Solution a. We compute $H(1) = \int_1^1 \frac{1}{t} \, dt = 0$. For every $x > 0$, the function $f(t) = \frac{1}{t}$ is continuous on the interval $[1, x]$ if $x \geq 1$ (or $[x, 1]$ if $x < 1$.) Thus, $f$ is also Riemann integrable on $[1, x]$ (similarly, $[1, x]$).

Furthermore, as $f$ is continuous at $x$ for all $x > 0$, by the Fundamental Theorem of Calculus (Theorem 7.5.1(ii) of Abbott or Theorem 12.6 of Hunter), $H$ is differentiable at $x$ and in fact, $H'(x) = f(x) = \frac{1}{x}$.

b. Let $x, y$ be arbitrary with $0 < x < y$.

Then by the additivity of integrals, we have

$$H(x) + \int_x^y \frac{1}{t} \, dt = \int_1^x \frac{1}{t} \, dt + \int_x^y \frac{1}{t} \, dt = \int_1^y \frac{1}{t} \, dt = H(y).$$

Since $f(t) = \frac{1}{t}$ is strictly positive for all $t > 0$, we have the inequality $\int_x^y \frac{1}{t} \, dt > 0$ for all $x, y$ with $0 < x < y$.

It follows that $H(x) < H(y)$, proving the desired property.

c. Fix $c > 0$ and consider the function $g$ given by $g(x) = H(cx)$.

As $H$ was differentiable and $H'(x) = \frac{1}{x}$ from part (a), using the Chain Rule, $g$ is differentiable at $x$ for all $x > 0$ and

$$g'(x) = c \cdot H'(cx) = c \cdot \frac{1}{cx} = \frac{1}{x} = H'(x)$$

As both $g$ and $H$ are differentiable at $x$ for all $x > 0$ and $g' = H'$, we conclude that there is a constant $k$ such that $H(cx) = g(x) = H(x) + k$.

Since $H(cx) = H(x) + k$, in particular for $x = 1$, we have $H(c) = H(1) + k = k$. Therefore, we have $H(cx) = H(x) + H(c)$ for all $x > 0$ and for all $c > 0$.
Problem 3 (Exercise 7.5.7: Average Value) If $g$ is a continuous on $[a, b]$, show that there exists a point $c \in (a, b)$ where

$$g(c) = \frac{1}{b - a} \int_a^b g.$$

Solution Define $G : [a, b] \to \mathbb{R}$ by $G(x) = \int_a^x g(t) \, dt$. Note that $G(a) = \int_a^a g(t) \, dt = 0$.

As $g$ is continuous on $[a, b]$, it is also Riemann integrable on $[a, b]$. By the Fundamental Theorem of Calculus, $G$ is continuous on $[a, b]$. Furthermore, as $g$ is continuous at $x$ for all $x \in [a, b]$, $G$ is differentiable on $[a, b]$ and $G'(x) = g(x)$.

By applying the Mean Value Theorem, there exists a point $c \in (a, b)$

$$g(c) = G'(c) = \frac{G(b) - G(a)}{b - a} = \frac{\int_a^b g(t) \, dt - 0}{b - a} = \frac{1}{b - a} \int_a^b g. \quad \text{wwu}$$