Problem 1 (Exercise 6.2.13) Review the construction of the Cantor set \( C \subseteq [0, 1] \) from Section 3.1. This exercise makes use of results and notation from this discussion.

Firstly, prove that the Cantor \( C \) has Lebesgue measure zero.

a. Define \( f_0(x) = x \) for all \( x \in [0, 1] \). Now, let
\[
f_1(x) = \begin{cases} (3/2)x, & \text{for } 0 \leq x \leq 1/3, \\ 1/2, & \text{for } 1/3 < x < 2/3, \\ (3/2)x - 1/2, & \text{for } 2/3 \leq x \leq 1. \end{cases}
\]

Sketch \( f_0 \) and \( f_1 \) over \([0, 1]\) and observe that \( f_1 \) is continuous, increasing and constant on the middle third \((1/3, 2/3) = [0, 1] \setminus C_1\).

b. Construct \( f_2 \) by imitating this process of flattening out the middle third of each nonconstant segment of \( f_1 \). Specifically, let
\[
f_2(x) = \begin{cases} (1/2)f_1(3x), & \text{for } 0 \leq x \leq 1/3, \\ f_1(x), & \text{for } 1/3 < x < 2/3, \\ 1/2f_1(3x - 2) + 1/2, & \text{for } 2/3 \leq x \leq 1. \end{cases}
\]

If we continue this process, show that the resulting sequence \( (f_n) \) converges uniformly on \([0, 1]\).

c. Let \( f = \lim f_n \). Prove that \( f \) is a continuous, increasing function on \([0, 1]\) with \( f(0) = 0 \) and \( f(1) = 1 \) that satisfies \( f'(x) = 0 \) for all \( x \) in the open set \([0, 1] \setminus C\). Recall that the “length” (Lebesgue measure) of the Cantor set \( C \) is 0. Somehow, \( f \) manages to increase from 0 to 1 while remaining constant on a set of “length 1” (Lebesgue measure 1).

Solution We prove that the Cantor set \( C \) has Lebesgue measure zero. Recall that \( C \) is constructed as (in Hunter’s notes)

\[ C = \bigcap_{n=1}^{\infty} F_n, \]

where \( F_n = \bigcup_{s \in \Sigma_n} I_s \) is a union of \( 2^n \) intervals \( I_s \), each of length \( 3^{-n} \), where \( s \) indexed by \( \Sigma_n \), the set of all binary sequences of length \( n \).

Let \( \varepsilon > 0 \) and choose \( N \) large enough so that \( 2 \left( \frac{2}{3} \right)^N < \varepsilon \). For each interval \( I_s \) in \( F_N \), choose open intervals \( J_s \) of length \( 2 \cdot 3^{-N} \) such that \( I_s \subseteq J_s \) (this is always possible as \( |I_s| = 3^{-N} < 2 \cdot 3^{-N} \)). Hence, we have

\[ C \subseteq F_N = \bigcup_{s \in \Sigma_N} I_s \subseteq \bigcup_{s \in \Sigma_N} J_s, \]

and that \( \sum_{s \in \Sigma_N} |J_s| = 2^N(2 \cdot 3^{-N}) = 2 \left( \frac{2}{3} \right)^N < \varepsilon \). This establishes that \( C \) has Lebesgue measure zero.

a. A sketch of \( f_0 \) is provided in Figure 1a whereas the sketch of \( f_1 \) is provided in Figure 1b.
b. We will begin by showing that for every integer \( n \geq 1 \),

\[
\sup_{x \in [0, 1]} |f_{n+1}(x) - f_n(x)| \leq \frac{1}{2} \sup_{y \in [0, 1]} |f_n(y) - f_{n-1}(y)|.
\]

For every function \( g : [0, 1] \to [0, 1] \), define a new function \( T(g) : [0, 1] \to [0, 1] \) as

\[
T(g)(x) = \begin{cases} 
\frac{1}{2} g(3x), & \text{if } 0 \leq x \leq \frac{1}{3}, \\
\frac{1}{2}, & \text{if } \frac{1}{3} < x < \frac{2}{3}, \\
\frac{1}{2} g(3x - 2) + \frac{1}{2}, & \text{if } \frac{2}{3} \leq x \leq 1,
\end{cases}
\]

Note that \( T(f_n) = f_{n+1} \) for all \( n \geq 0 \). We claim that for all \( g, h : [0, 1] \to [0, 1] \),

\[
\sup_{x \in [0, 1]} |T(g)(x) - T(h)(x)| \leq \frac{1}{2} \sup_{y \in [0, 1]} |g(y) - h(y)|.
\]

We break into different cases:

- For all \( 0 \leq x \leq \frac{1}{3} \), \( |T(g)(x) - T(h)(x)| = \left| \frac{1}{2} g(3x) - \frac{1}{2} h(3x) \right| \leq \frac{1}{2} \sup_{y \in [0, 1]} |g(y) - h(y)|. \)

- For all \( \frac{1}{3} < x < \frac{2}{3} \), \( |T(g)(x) - T(h)(x)| = \left| \frac{1}{2} - \frac{1}{2} \right| = 0 \leq \frac{1}{2} \sup_{y \in [0, 1]} |g(y) - h(y)|. \)

- For all \( \frac{2}{3} \leq x \leq 1 \), \( |T(g)(x) - T(h)(x)| = \frac{1}{2} \left| g(3x - 2) - h(3x - 2) \right| \leq \frac{1}{2} \sup_{y \in [0, 1]} |g(y) - h(y)|. \)

In all cases, we have \( |T(g)(x) - T(h)(x)| \leq \frac{1}{2} \sup_{y \in [0, 1]} |g(y) - h(y)| \), so by taking the supremum over all \( x \in [0, 1] \), we have the desired conclusion. Specializing to \( g = f_n \) and \( h = f_{n-1} \) for each \( n \geq 1 \), we may show inductively that for all \( n \geq 1 \),

\[
\sup_{x \in [0, 1]} |f_{n+1}(x) - f_n(x)| \leq \frac{1}{2} \sup_{y \in [0, 1]} |f_n(y) - f_{n-1}(y)|,
\]

and hence inductively, we have

\[
\sup_{x \in [0, 1]} |f_{n+1}(x) - f_n(x)| \leq \frac{1}{2^n} \sup_{y \in [0, 1]} |f_1(y) - f_0(y)|.
\]

In particular, by observing that \( 0 \leq f_0(y) \leq 1 \) and \( 0 \leq f_1(y) \leq 1 \) for all \( y \in [0, 1] \), we have the crude estimate that \( \sup_{y \in [0, 1]} |f_1(y) - f_0(y)| \leq 1 \) so that

\[
\sup_{x \in [0, 1]} |f_{n+1}(x) - f_n(x)| \leq \frac{1}{2^n}.
\]
Now, we use this estimate and Cauchy criterion for uniform convergence to prove that \((f_n)\) is a uniformly convergent sequence of functions on \([0, 1]\). Let \(\varepsilon > 0\) be arbitrary and choose an \(N\) such that \(\frac{1}{2N} < \varepsilon\).

Let \(m, n\) be integers such that \(m, n > N\); without loss of generality, assume that \(m < n\). Then, as \(\sum_{k=m}^{\infty} \frac{1}{2^k}\) converges to \(\frac{1}{2^{m-1}}\), we have

\[
|f_n(x) - f_m(x)| \leq \sum_{k=m}^{n-1} |f_{k+1}(x) - f_k(x)|
\leq \sum_{k=m}^{\infty} |f_{k+1}(x) - f_k(x)|
\leq \sum_{k=m}^{\infty} \frac{1}{2^k}
\leq \frac{1}{2^{m-1}}
\leq \frac{1}{2N}
\leq \varepsilon.
\]

It follows from Cauchy criterion that \((f_n)\) converges uniformly on \([0, 1]\) (without having to know what \(f\) actually is).

c. Since it was proven that \((f_n)\) converges uniformly on \([0, 1]\) in part (b) and uniform convergence implies pointwise convergence, \(f_n \to f\) uniformly on \([0, 1]\).

Using the transformation \(T\) defined as above, we note that for all functions \(g : [0, 1] \to [0, 1]\), if \(g\) is continuous and \(g(0) = 0\) and \(g(1) = 1\), then \(T(g)\) is continuous because:

- The functions \(\frac{1}{2}g(3x)\), \(\frac{1}{2}\), \(\frac{1}{2}g(3x - 2) + \frac{1}{2}\) are compositions and sums of continuous functions, so are continuous on the interior of the respective intervals they are defined.
- \(\lim_{x \to 1/3^-} T(g)(x) = T(g)(1/3) = \frac{1}{2}g(1) = \frac{1}{2} = \lim_{x \to 1/3^-} T(g)(x)\).
- \(\lim_{x \to 2/3^+} T(g)(x) = T(g)(2/3) = \frac{1}{2}g(0) + \frac{1}{2} = \frac{1}{2} = \lim_{x \to 2/3^-} T(g)(x)\).

Furthermore, \(T(g)(0) = \frac{1}{2}g(0) = 0\) and \(T(g)(1) = \frac{1}{2}g(1) + \frac{1}{2} = 1\) under the same setting. We also note that if \(g : [0, 1] \to [0, 1]\) is increasing, then \(T(g)(x) \leq \frac{1}{2} \leq T(g)(y)\) for all \(x \in [0, 1/3]\) and for all \(y \leq 2/3, 1\). As a consequence, by inspection on the separate intervals, \(T(g)\) is also increasing.

By noting that \(f_0(x) = x\) is continuous and increasing on \([0, 1]\), \(f_0(0) = 0\), \(f_0(1) = 1\), and that \(T(f_n) = f_{n+1}\) for all integers \(n \geq 0\), inductively on \(n\), we may conclude that each \(f_n\) is continuous and increasing on \([0, 1]\) with \(f_n(0) = 0\) and \(f_n(1) = 1\).

As uniform convergence preserve continuity and monotonicity, and that in particular, each \(f_n\) is continuous and increasing on \([0, 1]\), it follows that \(f\) is continuous and increasing on \([0, 1]\). Moreover, \(f_n(0) = 0\) and \(f_n(1) = 1\) for each \(n\), so we obtain \(f(0) = \lim_{n \to \infty} f_n(0) = 0\) and \(f(1) = \lim_{n \to \infty} f_n(1) = 1\).

If \(x \in [0, 1]\) \(\setminus C\), then \(x \notin C = \bigcap_{n=1}^{\infty} F_n\), so that \(x \notin F_n\) for some \(n \geq 1\). This means that there is an \(n \geq 1\) and an open interval \(U \subset [0, 1] \setminus F_n\) containing \(x\) that was a middle third of some closed interval in \(F_{n-1}\) (here, we take the convention that \(F_0 = [0, 1]\)). Thus, using the recursive definition for \(f_{k+1}\) in terms of \(f_k\), there is a \(0 \leq c \leq 1\) such that \(f_k(y) = c\) for all \(y \in U\) and for all \(k \geq n\). In other words, \((f_k)\) is eventually constant on \(U\), so that \(f(y) = \lim_{k \to \infty} f_k(y) = c\) is a constant for all \(y \in U\). In particular, we have \(f'(x) = 0\).
Problem 2 (Exercise 6.3.3) Consider the sequence of functions

\[ f_n(x) = \frac{x}{1+nx^2}. \]

Exercise 6.2.4 contains some advice for how to show that \( (f_n) \) converges uniformly on \( \mathbb{R} \). Review or complete this exercise.

Now, let \( f = \lim f_n \). Compute \( f_n'(x) \) and find all values of \( x \) for which \( f'(x) = \lim f_n'(x) \).

Solution From Exercise 6.2.4 in HW 15, \( f_n \to 0 \) uniformly on \( \mathbb{R} \). Thus, \( f \equiv 0 \) on \( \mathbb{R} \) and \( f' \equiv 0 \).

Now, \( f_n'(x) = \frac{(1+nx^2)(1) - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2} \). For each fixed nonzero \( x \in \mathbb{R} \), we compute

\[
\lim_{n \to \infty} f_n'(x) = \lim_{n \to \infty} \frac{1-nx^2}{(1+nx^2)^2} \\
= \lim_{n \to \infty} \frac{1/n^2 - x^2/n}{(1/n + x^2)^2} \\
= 0 - 0 \\
= \frac{0}{(0+x^2)^2} \\
= 0 \\
= f'(x).
\]

On the other hand, \( f_n'(0) = 1 \) for each positive integer \( n \), so that \( \lim_{n \to \infty} f_n'(0) = 1 \neq 0 = f'(0) \). Therefore, \( f'(x) = \lim f_n'(x) \) if and only if \( x \neq 0 \).

Problem 3 (Exercise 7.4.1) a. Let \( f \) be a bounded function on a set \( A \), and set

\[
M = \sup \{ f(x) : x \in A \}, \quad m = \inf \{ f(x) : x \in A \},
\]

\[
M' = \sup \{ |f(x)| : x \in A \}, \quad m' = \inf \{ |f(x)| : x \in A \}.
\]

Show that \( M - m \geq M' - m' \).

b. Show that if \( f \) is integrable on the interval \( [a,b] \), then \( |f| \) is also integrable on this interval.

c. Provide the details for the argument that in this case we have

\[
\left| \int_a^b f \right| \leq \int_a^b |f|.
\]

Solution a. Using the triangle inequality, for all \( x,y \in A \), we have both

\[
|f(x)| \leq |f(x) - f(y)| + |f(y)|, \quad |f(y)| \leq |f(x) - f(y)| + |f(x)|,
\]

which imply that

\[
-|f(x) - f(y)| \leq |f(x)| - |f(y)| \leq |f(x) - f(y)|.
\]

Now, \( f(x) \leq M \) for all \( x \in A \) and \( f(y) \geq m \) for all \( y \in A \). Hence, \( f(x) - f(y) \leq M - m \) for all \( x,y \in A \). A similar argument shows that \( -(M-m) \leq f(y) - f(x) \) for all \( x,y \in A \), so that \( |f(z) - f(z')| \leq |M-m| \) for all \( z, z' \in A \).

In particular, \( M - m \geq |f(x)| - |f(y)| \) for all \( x,y \in A \). Fixing \( y \in A \) and taking the supremum of \( |f(x)| \) over all \( x \in A \) yields \( M - m \geq M' - |f(y)| \), while using this and taking the infimum of \( |f(y)| \) over all \( y \in A \) yields \( M - m \geq M' - m' \), proving the required result.

b. Using the Cauchy criterion, it is enough to show that for every \( \varepsilon > 0 \), there is a partition \( P \) with \( U(|f|;P) - L(|f|;P) < \varepsilon \).

Suppose that \( f \) is Riemann integrable on \([a,b]\). Let \( \varepsilon > 0 \). Then, by the Cauchy criterion, we have that for every \( \varepsilon > 0 \), there is a partition \( P' \) with \( U(f;P') - L(f;P') < \varepsilon \).
Set $P = P'$ and write $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $a = x_0 < x_1 < x_2 < \ldots < x_n = b$. For each $1 \leq i \leq n$, denote
\[
m_i = \inf_{y \in [x_{i-1}, x_i]} f(y), \quad M_i = \sup_{y \in [x_{i-1}, x_i]} f(y),
\]
\[
m_i' = \inf_{y \in [x_{i-1}, x_i]} |f(y)|, \quad M_i' = \sup_{y \in [x_{i-1}, x_i]} |f(y)|.
\]
Using the result in part (a), we note that $M_i' - m_i' \leq M_i - m_i$ for each $1 \leq i \leq n$. Thus,
\[
U(|f|; P) - L(|f|; P) = \sum_{i=1}^{n} M_{i}'(x_i - x_{i-1}) - \sum_{i=1}^{n} m_{i}'(x_i - x_{i-1})
= \sum_{i=1}^{n} (M_{i}' - m_{i}') (x_i - x_{i-1})
\leq \sum_{i=1}^{n} (M_{i} - m_{i}) (x_i - x_{i-1})
= \sum_{i=1}^{n} M_{i}(x_i - x_{i-1}) - \sum_{i=1}^{n} m_{i}(x_i - x_{i-1})
= U(f; P) - L(f; P)
< \varepsilon.
\]
Therefore, $|f|$ is Riemann integrable on $[a, b]$ too.

c. In the case when $f$ is Riemann integrable on $[a, b]$, part (b) implies that $|f|$ is Riemann integrable on $[a, b]$, so that both $\int_{a}^{b} f(x) \, dx$ and $\int_{a}^{b} |f(x)| \, dx$ are defined.

For all $x \in [a, b]$, we have $-|f(x)| \leq f(x) \leq |f(x)|$. Therefore, as integrals preserve monotonicity, we may conclude that
\[
- \int_{a}^{b} |f(x)| \, dx \leq \int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} |f(x)| \, dx.
\]
It follows that
\[
\left| \int_{a}^{b} f(x) \, dx \right| \leq \int_{a}^{b} |f(x)| \, dx.
\]